WELL-CHAINED RELATOR SPACES

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Introduction

In this paper, we extend four basic characterizations of well-chained uniformities of Levine [4] to those of well-chained relators.

And combining our present results with some former ones, we establish some substantial generalizations of two relevant theorems of Gaal [1, pp. 101 and 142].

The necessary prerequisites concerning relators, which are possibly unfamiliar to the reader, will be briefly laid out in the next two preparatory sections.

0. Terminology and notations

If \( \mathcal{R} \) is a nonvoid family of reflexive relations \( R \) on a set \( X \), the family \( \mathcal{R} \) is called a relator on \( X \), and the ordered pair \( X(\mathcal{R}) = (X, \mathcal{R}) \) is called a relator space.

If \( (x_\alpha) \) and \( (y_\alpha) \) are nets, \( A \) and \( B \) are sets, and \( x \) is a point in a relator space \( X(\mathcal{R}) \), then we write

(i) \( (y_\alpha) \in \text{Lim}_R(x_\alpha) \) \( ((y_\alpha) \in \text{Adh}_R(x_\alpha)) \) if \( ((x_\alpha, y_\alpha)) \) is eventually (frequently) in each \( R \in \mathcal{R} \);

(ii) \( x \in \text{lim}_R(x_\alpha)(x \in \text{adh}_R(x_\alpha)) \) if \( (x) \in \text{Lim}_R(x_\alpha) \) \( ((x) \in \text{Adh}_R(x_\alpha)) \);

(iii) \( B \in \text{Cl}_R(A) \) \( (B \in \text{Int}_R(A)) \) if \( R(B) \cap A \neq \emptyset \) \( (R(B) \subseteq A) \) for all (some) \( R \in \mathcal{R} \);

(iv) \( x \in \text{cl}_R(A)(x \in \text{int}_R(A)) \) if \( x \in \text{Cl}_R(A) \) \( (x \in \text{Int}_R(A)) \).

If \( \mathcal{R} \) is a relator on \( X \), then the relators

\[
\mathcal{R}^* = \{ S \subseteq X^2 : \exists R \in \mathcal{R} : R \subseteq S \},
\]

\[
\mathcal{R}^f = \{ S \subseteq X^2 : \forall A \subseteq X : \exists R \in \mathcal{R} : R(A) \subseteq S(A) \},
\]

\[
\mathcal{R} = \{ S \subseteq X^2 : \forall x \in X : \exists R \in \mathcal{R} : R(x) \subseteq S(x) \},
\]

Received November 20, 1989.
are called the uniform, proximal and topological refinements of \( \mathcal{R} \), respectively.

Namely, if \( \mathcal{R} \) is a relator on \( X \), then \( \mathcal{R}^* \), \( \mathcal{R}^\# \) and \( \hat{\mathcal{R}} \) are the largest relators on \( X \) such that \( \lim_{\mathcal{R}^*} = \lim_{\mathcal{R}}(\text{Adh}_{\mathcal{R}^*} = \text{Adh}_{\mathcal{R}}) \), \( \text{Cl}_{\mathcal{R}^*} = \text{Cl}_{\mathcal{R}} \) \((\text{Int}_{\mathcal{R}^*} = \text{Int}_{\mathcal{R}}) \) and \( \lim_{\hat{\mathcal{R}}} = \lim_{\mathcal{R}} \left( \text{adh}_{\hat{\mathcal{R}}} = \text{adh}_{\mathcal{R}} \right) \) or \( \text{cl}_{\hat{\mathcal{R}}} = \text{cl}_{\mathcal{R}} \left( \text{int}_{\hat{\mathcal{R}}} = \text{int}_{\mathcal{R}} \right) \), respectively.

Moreover, a subset \( A \) of a relator space \( X(\mathcal{R}) \) is called
(i) proximally closed (open) if \( X \setminus A \not\subseteq \text{Cl}_{\mathcal{R}}(A)(A \in \text{Int}_{\mathcal{R}}(A)) \);
(ii) topologically closed (open) if \( \text{cl}_{\mathcal{R}}(A) \subseteq A(A \subseteq \text{int}_{\mathcal{R}}(A)) \);
(iii) proximally (topologically) clopen if it is both proximally (topologically) closed and open.

Clearly, a proximally closed (open) set is also topologically closed (open), but the converse need not be true. Moreover, a set is proximally (topologically) closed iff its complement is proximally (topologically) open.

On the other hand, a relator \( \mathcal{R} \) on \( X \) or a relator space \( X(\mathcal{R}) \) is called topologically compact if for each \( R \in \mathcal{R} \) there exists a finite set \( A \subseteq X \) such that \( R(A) = X \).

Namely, a relator space \( X(\mathcal{R}) \) is topologically compact iff each interior cover \( \mathcal{A} \) of \( X(\mathcal{R}) \) has a finite subcover \( \mathcal{B} \), or equivalently each directed net \( (x_\alpha) \) in \( X(\mathcal{R}) \) is adherent.

Finally, a relator \( \mathcal{R} \) on \( X \), or a relator space \( X(\mathcal{R}) \), is called
(i) uniformly directed if for each \( R, S \in \mathcal{R} \) there exists a \( T \in \mathcal{R} \) such that \( T \subseteq R \cap S \);
(ii) strongly proximally directed if for any \( A_i \subseteq X \) and \( R_i \in \mathcal{R} \) with \( i = 1, 2, \ldots, n \), there exists an \( R \in \mathcal{R} \) such that \( R(A_i) \subseteq R_i(A_i) \) for all \( i = 1, 2, \ldots, n \).
(iii) topologically transitive if for each \( x \in X \) and \( R \in \mathcal{R} \) there exist \( S, T \in \mathcal{R} \) such that \( T(S(x)) \subseteq R(x) \);
(iv) proximally symmetric if for each \( A \subseteq X \) and \( R \in \mathcal{R} \) there exists an \( S \in \mathcal{R} \) such that \( S(A) \subseteq R^{-1}(A) \).

Clearly, a uniformly directed relator is also strongly proximally directed, but the converse need not be true. On the other hand, a relator \( \mathcal{R} \) is proximally symmetric iff the relation \( \text{Cl}_{\mathcal{R}} \) is symmetric.

1. Some basic facts on connected relators

Definition 1.1. A relator \( \mathcal{R} \) on \( X \), or a relator space \( X(\mathcal{R}) \) is called connected if \( A^2 \cup (X \setminus A)^2 \not\subseteq \mathcal{R} \) for all proper nonvoid subset \( A \) of \( X \).

Moreover, \( \mathcal{R} \) or \( X(\mathcal{R}) \) is called uniformly, proximally and topologically
connected if the relators $\mathcal{R}^*, \mathcal{R}^\sharp$ and $\mathcal{R}$ are connected, respectively.

The appropriateness of this definition and the validity of the next theorems have been established in [3].

**Theorem 1.2.** A relator space $X(\mathcal{R})$ is proximally (topologically) connected if no proper nonvoid subset $A$ of $X(\mathcal{R})$ is proximally (topologically) clopen.

**Theorem 1.3.** A proximally symmetric relator space $X(\mathcal{R})$ is proximally connected if no proper nonvoid subset $A$ of $X(\mathcal{R})$ is proximally open.

**Theorem 1.4.** A proximally symmetric and topologically fine relator space $X(\mathcal{R})$ is topologically connected if and only if no proper nonvoid subset $A$ of $X(\mathcal{R})$ is topologically open.

**Theorem 1.5.** A uniformly directed relator space $X(\mathcal{R})$ is proximally connected if and only if it is uniformly connected.

To state a further relevant property of connected relators, we also need to the following.

**Definition 1.6.** A relator $\mathcal{R}$ on $X$ is called a Lebesgue relator, and a relator space $X(\mathcal{R})$ is called a Lebesgue relator space if for each $S \in \mathcal{R}$ there exists a function $f$ from $X$ into $X$ such that $S \circ f \in \mathcal{R}$.

The appropriateness of this definition and the validity of the next theorem have been established in [8].

**Theorem 1.7.** A strongly proximally directed, topologically transitive and topologically compact relator space $X(\mathcal{R})$ is a Lebesgue relator space.

Moreover, as a particular case of a more general result, we also have

**Theorem 1.8.** A Lebesgue relator space $X(\mathcal{R})$ is topologically connected if and only if it is uniformly connected.

### 2. Preliminary characterizations of well-chained relators

The origin of the following definition goes back to Cantor. (See Thron[9, p.29].)

**Definition 2.1.** A relator $\mathcal{R}$ on $X$, or a relator space $X(\mathcal{R})$, will be called well-chained if for any $x, y \in X$ and $R \in \mathcal{R}$ there exists a finite family $(x_i)_{i=0}^n$ in $X$ such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all
Moreover, $\mathcal{R}$ or $X(\mathcal{R})$ will be called uniformly, proximally and topologically well-chained if the relators $\mathcal{R}^*, \mathcal{R}^\dagger$ and $\hat{\mathcal{R}}$ are well-chained, respectively.

Remark 2.2. Because of the inclusions $\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\dagger \subset \hat{\mathcal{R}}$, it is clear that ‘topologically well-chained’ $\Rightarrow$ ‘proximally well-chained’ $\Rightarrow$ ‘uniformly well-chained’ $\Rightarrow$ ‘well-chained’.

In the sequel, we shall show that ‘uniformly well-chained’ and ‘proximally well-chained’ are actually equivalent to ‘well-chained’, but ‘topologically well-chained’ is not equivalent to ‘well-chained’.

For this, we shall first extend three basic characterizations of well-chained uniformities of Levine [4] to those of well-chained relators.

Our first theorem is a straightforward extension of Levine’s [4, Corollary 2.3].

**Theorem 2.3.** If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:

(i) $X(\mathcal{R})$ is well-chained;

(ii) $X^2 = \bigcup_{n=1}^{\infty} R^n$ for all $R \in \mathcal{R}$.

**Proof.** A simple reformulation of Definition 2.1 shows that (i) holds if and only if for any $x, y \in X$ and $R \in \mathcal{R}$ there exists a positive integer $n$ such that $(x, y) \in R^n$. And hence, the equivalence of (i) and (ii) is quite obvious.

While, our second theorem is a natural extension of Levine’s [4, Theorem 2.2].

**Theorem 2.4.** If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:

(i) $X(\mathcal{R})$ is well-chained;

(ii) $X^2$ is the only transitive member of $\mathcal{R}^*$.

**Proof.** If $S \in \mathcal{R}^*$, then there exists an $R \in \mathcal{R}$ such that $R \subset S$. Therefore, if (i) holds and $S$ is transitive, then by Theorem 2.3 we clearly have

$$X^2 = \bigcup_{n=1}^{\infty} R^n \subset \bigcup_{n=1}^{\infty} S^n \subset S.$$ 

And thus (ii) also holds.

On the other hand, if $R \in \mathcal{R}$, then it is clear that

$$S = \bigcup_{n=1}^{\infty} R^n$$
is a transitive relation on $X$ such that $R \subseteq S$. Therefore, if (ii) holds, then we necessarily have $S = X^2$. Thus, again by Theorem 2.3, (i) also holds.

Remark 2.5. Because of the reflexivity of the elements of $\mathcal{R}$, hence we can also state that $X(\mathcal{R})$ is well-chained if and only if $X^2$ is the only preorder in $\mathcal{R}^*$.

3. Main characterizations of well-chained relators

Now, having Theorem 2.3 and 2.4, we can also easily prove a natural extension of Levine's [4, Corollary 2.4].

Theorem 3.1. If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:

(i) $X(\mathcal{R})$ is well-chained;

(ii) no proper nonvoid subset $A$ of $X(\mathcal{R})$ is proximally open.

Proof. If (ii) does not hold, then there exists a proper nonvoid subset $A$ of $X$ such that $R(A) \subset A$ for some $R \in \mathcal{R}$. Hence, it is clear that

$$(\bigcup_{n=1}^{\infty} R^n)(A) = \bigcup_{n=1}^{\infty} R^n(A) \subset A.$$ 

And thus, by Theorem 2.3, (i) does not also hold. Consequently, (i) implies (ii).

On the other hand, if (i) does not hold, then by Theorem 2.4, there exists a transitive relation $S$ on $X$ such that $R \subseteq S$ for some $R \in \mathcal{R}$, and $A = S(x) \neq X$ for some $x \in X$. Hence, it is clear that

$$R(A) \subset S(A) = S^2(x) = S(x) = A.$$ 

And thus (ii) does not also hold. Consequently, (ii) also implies (i).

Remark 3.2. Because of [6, Theorem 2.6], hence we can also state that a relator space $X(\mathcal{R})$ is well-chained if and only if no proper nonvoid subset $A$ of $X(\mathcal{R})$ is proximally closed.

Remark 3.3. Moreover, by [6, Theorem 3.1], hence we can also state that a relator space $X(\mathcal{R})$ is well-chained if and only if for each proper nonvoid subset $A$ of $X$ there exists a net $((x_\alpha, y_\alpha))$ in $A \times (X \setminus A)$ such that $(x_\alpha) \in \text{Lim}_{(y_\alpha)} R(x_\alpha)$, $(x_\alpha) \in \text{Adh}_{(y_\alpha)} R(x_\alpha)$.

However, at the present, it is more interesting to point out that Theorem 3.1 can also be used to easily prove the next two important theorems.
Theorem 3.4. If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:

(i) $X(\mathcal{R})$ is well-chained;
(ii) $X(\mathcal{R})$ is uniformly well-chained;
(iii) $X(\mathcal{R})$ is proximally well-chained.

Proof. By [5, Corollary 5.9], it is clear that the proximally open subsets of $X(\mathcal{R}^i)$ and $X(\mathcal{R}^*)$ coincide with those of $X(\mathcal{R})$. And thus Theorem 3.1 can be applied to get the stated equivalences.

Theorem 3.5. If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:

(i) $X(\mathcal{R})$ is topologically well-chained;
(ii) no proper nonvoid subset $A$ of $X(\mathcal{R})$ is topologically open.

Proof. By [5, Theorem 6.7], it is clear that the proximally open subsets of $X(\mathcal{R})$ coincide with the topologically open subsets of $X(\mathcal{R})$. Thus, Theorem 3.1 can again be applied to get the stated equivalence.

The fact that ‘topologically well-chained’ is not, in general, equivalent to ‘well-chained’ can be at once seen from the next simple.

Example 3.6. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for $i = 1, 2$, such that

$$R_1(1) = \{1, 2\}, \quad R_1(2) = \{2, 3\}, \quad R_1(3) = \{3, 1\},$$
$$R_2(1) = \{1, 2\}, \quad R_2(2) = X, \quad R_2(3) = \{3, 2\},$$

then $\mathcal{R} = \{R_i\}^2_{i=1}$ is a well-chained relator on $X$ such that $\mathcal{R}$ is not topologically well-chained.

To check this, note that $R_i^2 = X^2$ for $i = 1, 2$. And moreover, if $S \subset X^2$ such that

$$S(1) = \{1, 2\}, \quad S(2) = \{2, 3\}, \quad S(3) = \{3, 2\},$$

then $S \in \hat{\mathcal{R}}$, but $1 \notin S^n(2)$ for all positive integer $n$.

4. A further characterization of well-chained relators

In addition to the above theorems, using Theorem 2.4 and 3.1, we can also prove the following remarkable analogue of Levine’s [4, Corollary 2.5].

Theorem 4.1. If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:
(i) $X(\mathcal{R})$ is well-chained;
(ii) $A^2 \cup (X\setminus A) \times X \notin \mathcal{R}^*$ for all proper nonvoid subset $A$ of $X$.

Proof. If $A$ is a proper nonvoid subset of $X$, then it is clear that

$$S = A^2 \cup (X\setminus A) \times X$$

is a transitive relation on $X$ such that $S = X^2$. Therefore, if (i) holds, then by Theorem 2.4, we necessarily have $S \notin \mathcal{R}^*$. And thus (ii) also holds.

On the other hand, if (i) does not hold, then by Theorem 3.1, there exists a proper nonvoid subset $A$ of $X$ such that $R(A) \subset A$ for some $R \in \mathcal{R}$. Hence, it is clear that

$$R \subset A^2 \cup (X\setminus A) \times X,$$

and thus (ii) does not also hold. Consequently, (ii) also implies (i).

By this theorem, it is clear that the Davis-Pervin relator [6, p.195] cannot, in general, be well-chained. More precisely, using Theorem 4.1, one can easily check the next striking

**Example 4.2.** If $\mathcal{A}$ is a nonvoid family of subsets of a set $X$ and

$$\mathcal{R}_\mathcal{A} = \{A^2 \cup (X\setminus A) \times X : A \in \mathcal{A}\},$$

then the following assertions are equivalent:
(i) $\mathcal{R}_\mathcal{A}$ is well-chained;
(ii) $\mathcal{A} \subset \{\emptyset, X\}$;
(iii) $\mathcal{R}_\mathcal{A} = \{X^2\}$;
(iv) $\mathcal{R}_\mathcal{A}$ is topologically well-chained.

**Remark 4.3.** Using Theorem 1.3, in [3] we have proved that (i) $\mathcal{R}_\mathcal{A}$ is proximally connected if and only if there is no proper nonvoid subset $B$ of $X$ such that both $B$ and $X \setminus B$ are in $\mathcal{A}$;
(ii) $\mathcal{R}_\mathcal{A}$ is topologically connected if and only if there is no proper nonvoid subset $B$ of $X$ such that both $B$ and $X \setminus B$ are unions of certain members of $\mathcal{A}$.

5. Relationships with connected relators

As an immediate consequence of Theorem 1.2 and Theorems 3.1, 3.4 and 3.5, we can at once state
Theorem 5.1. A proximally (topologically) well-chained relator space $X(\mathcal{R})$ is proximally (topologically) connected.

Hence, by Theorem 1.8, it is clear that we also have the following useful.

Theorem 5.2. A well-chained Lebesgue relator space $X(\mathcal{R})$ is topologically connected.

This latter theorem, together with Theorem 1.7, at once yields a substantial extension of the ‘if part’ of Theorem III.3.9 of Gaal [1, p.142].

Theorem 5.3. A strongly proximally directed, topologically transitive and topologically compact well-chained relator space $X(\mathcal{R})$ is topologically connected.

Remark 5.4. By example 4.2 and Remark 4.3, it is clear that even a topologically connected relator space $X(\mathcal{R})$ need not be well-chained.

However, combining Theorem 1.3 with Theorems 3.1 and 3.4, we can still state an essential improvement of Theorem II.7.3 of Gaal [1, p. 101].

Theorem 5.5. A proximally symmetric relator space $X(\mathcal{R})$ is proximally well-chained if and only if it is proximally connected.

Hence, by Theorem 1.5 and Theorem 3.4, it is clear that we also have the following extension of Levine’s [4, Corollary 2.5].

Theorem 5.6. A uniformly directed and proximally symmetric relator space $X(\mathcal{R})$ is uniformly well-chained if and only if it is uniformly connected.

Moreover, as an immediate consequence of Theorem 1.4 and Theorem 3.5, we can also state the following analogue of Levine’s [4, Corollary 4.2].

Theorem 5.7. A topologically fine and proximally symmetric relator space $X(\mathcal{R})$ is topologically well-chained if and only if it is topologically connected.

Notes. Particular cases of Theorems 5.3 and 5.5 are also treated in Whyburn and Duda [10, p.37].

Moreover, a slightly incorrect particular case of Theorem 5.6 can also be found in James [2, p. 126].
References


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