SOME TOPOLOGICAL OPERATORS VIA IDEALS

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An ideal on a nonempty set $X$ is a collection $I$ of subsets of $X$ which is closed under the operations of subset (heredity) and finite unions (additivity). In this paper, we introduce and study three different notions via ideals namely, semi local functions, the set operator $\Psi^s(I, T)$ and semi compatibility of $T$ with $I$. We characterize these new sorts. Several properties of them have been studied. Also, their relationship with other types of operators are investigated.

1. Introduction

Throughout this paper, $(X, T)$ is a topological space on which no separation axioms are assumed unless explicitly stated. The notation $(X, T, I)$ will denote a topological space $(X, T)$ and an ideal $I$ on $X$ with no separation properties assumed. For $A \subseteq (X, T), Cl(A)$ and $int(A)$, respectively, denote the closure and the interior of $A$ with respect to $T$, “neighborhood” will be abbreviated “nbd”, “iff” means “if and only if”, and $N(x)$ denotes the open neighbourhood system at a point $x \in X$, i.e. $N(x) = \{U \in T : x \in U\}$. $P(A)$, the power set of $A$, is the set of all subsets of $A$. $A \subseteq X$ is called semi-open [8], if $A \subseteq Cl(int(A))$. The complement of semi open set is called semi closed set [3]. The intersection (resp. union) of all semi-closed (resp. semi-open) sets that contain (resp. contained in ) $A$ is called the semi-closure [2] (resp. semi-interior [2]) of $A$ and is denoted by $s-CI(A)$ (resp. $s-int(A)$). A subset $N_x \subseteq X$ is called a semi-neighbourhood (s.nbd) [12], of a point $x \in X$ if there exists a semi open set $A \subseteq X$, such that $x \in A \subseteq N_x$. The family of all semi-open sets (resp. semi-neighbourhood) of $X$ will be denoted by $SO(X)$ (resp.
A nonempty collection $I$ of subsets of $X$ is said to be an ideal [4] on $X$, if it satisfies the following two conditions:

1. If $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity),
2. If $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity).

Given a topological space $(X, \tau)$ and an ideal $I$ on $X$, a set operator $(\cdot)^{(I, \tau)} : P(X) \to P(X)$ called the local function of $I$ with respect to $\tau$ in [13], [6] is defined as follows: for $A \subseteq X$, $(A)^{(I, \tau)} = \{x \in X : U \cap A \notin I$ for every $U \in N(x)\}$. A Kuratowski closure operator $Cl^{(I)}$ for a topology $T^{(I)}$ finer than $\tau$ is defined as follows: $Cl^{(I)}(A) = A \cup (A)^{(I, \tau)}$ [13]. When there is no chance for confusion, we will simple write $A^*$ for $(A)^{(I, \tau)}$. A basis $\beta(I, \tau)$ for $T^{(I)}$ can be described as follows: $\beta(I, \tau) = \{U \in \mathcal{E} : U \in T, E \in I\}$ [4].

Remark 1.1. Many basic properties, and useful facts concerning the local functions, see [6].

In [9], Natkaniec defines an operator $\Psi : p(X) \to \tau$ where $(X, \tau, I)$ is a space as follows: for every $A \subseteq X$, $\Psi(A) = \{x : \text{there exists a } U \in N(x) \text{ such that } U \setminus A \in I\}$, and observes that $\Psi(A) = X \setminus (X \setminus A)^*$, see [5].

Remark 1.2. The operator $\Psi$ has been studied in [4] where the following is observed for every $A \subseteq X$. $\Psi(A) = \cup[U \in T : U \setminus A \in I]$ and $\Psi(A)$ is open.

Given a space $(X, \tau, I)$, Njastad [11] defines the ideal $I$ to be compatible with $\tau$, denoted by $I \sim \tau$, if for every $A \subseteq X$ and for every $x \in A$ there exists a $U \in N(x)$ such that $U \cap A \in I$, then $A \in I$ [5].

Remark 1.3. For several characterizations of Compatibility, see [6].

2. Semi local functions

In this article, we introduce new class of the set operator $(\cdot)^*$ utilizing on semi open neighbourhood namely the set operator $(\cdot)^{**}$.

Definition 2.1. Given a space $(X, \tau, I)$, a set operator $(\cdot)^{**} : P(X) \to P(X)$, called the semi local function of $I$ with respect to $\tau$, is defined as follows; for $A \subseteq X$, $(A)^{**}(I, \tau) = \{x \in X : U_x \cap A \notin I$, for every $U_x \in SN(x)\}$, where $SN(x) = \{U \in SO(X) : x \in U\}$. When there is no ambiguity, we will simply write $A^{**}(I)$ or $A^{**}$ for $(A)^{**}(I, \tau)$.

Remark 2.1.
(i) Each semi local function is local function but the converse is not true, in general, as shown by Example (2.1).
(ii) The collection of all semi open subsets of a space \((X, T)\) fails to form an ideal on \(X\) as shown by Example (2.2).

**Example 2.1.** Let \(X = \{a, b, c\}, T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \(I = \{\emptyset, \{a\}\}\). Then \(\{b\}^* = \{b, c\}\), but \(\{b\}^{**} = \{b\}\).

**Example 2.2.** Let \(X = \{a, b, c\}\) and \(T = \{\emptyset, X, \{a\}, \{a, b\}\}\). We notice that \(SO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}\). Then the operations of subset (heredity) does not satisfied.

If \((X, T, I)\) is a space, we denote by \(T^{**}(I)\) the topology on \(X\) generated by the subbasis \(\{U \setminus E : U \in SO(X) \text{ and } E \in I\}\).

The closure operator in \(T^{**}(I)\), denoted by \(Cl^{**}\), can be described as follows: for \(A \subseteq X\), \(Cl^{**}(A) = A \cup A^{**}(I)\).

**Remark 2.2.** We can easily deduce that \(T \subseteq T^{*}(I) \subseteq T^{**}(I)\) and the converse does not hold as the following example shows.

**Example 2.3.** Consider the topology \(T = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}\) on a set \(X = \{a, b, c, d\}\) and \(I = \{\emptyset, \{a\}\}\). Then \(T^{*}(I) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}\), and \(T^{**}(I) = P(X)\).

**Lemma 2.1.** \(T^{**}(I) = \{U \subseteq X : Cl^{**}(X \setminus U) = X \setminus U\}\).

**Corollary 2.1.** For \(A \subseteq (X, T, I)\), we have:

(i) If \(I = \{\emptyset\}\), then \(A^{**}(\{\emptyset\}) = s-Cl(A)\), and \(Cl^{**}(A) = s-Cl(A)\).

(ii) If \(I = P(X)\), then \(A^{**}(P(X)) = \emptyset\), and \(Cl^{**}(A) = A\). Hence in this case \(T^{**}(I)\) is the discrete topology.

**Proof.** (i) Follows directly from the definition 2.1 and the definition of semi closure point.

(ii) is obvious.

Several basic facts concerning the behaviour of the operator \((\cdot)^{**}\) are included in the following theorem.

**Theorem 2.1.** Let \((X, T, I)\) be a space and \(A, B \subseteq X\), then the following statements hold:

(i) \(\phi^{**}(I) = \phi\),

(ii) if \(A \subseteq B\), then \(A^{**}(I) \subseteq B^{**}(I)\),

(iii) \((A^{**})^{**}(I) \subseteq A^{**}(I)\),

(iv) \((A \cup B)^{**}(I) = A^{**}(I) \cup B^{**}(I)\),

(v) \((A \cap B)^{**}(I) \subseteq A^{**}(I) \cap B^{**}(I)\),

(vi) \(A^{**}(I) \setminus B^{**}(I) \subseteq (A \setminus B)^{**}(I)\).
(vii) \(A^{ss}(I) = s-Cl(A^{ss}(I)) \subseteq s-Cl(A)(A^{ss} \text{ is a semi closed subset of } s-Cl(A))\),

(viii) if \(E \in I\), then \((A \cup E)^{ss}(I) = A^{ss}(I) = (A \setminus E)^{ss}(I)\)

(ix) if \(U \in SO(X)\), then: \(U \cap A^{ss}(I) = U \cap (U \cap A)^{ss}(I) \subseteq (U \cap A)^{ss}(I)\).

**Proof.**

(i) Obvious, since \(\emptyset\) always belongs to \(I\).

(ii) This is an immediate consequence of the definition of semi local function.

(iii) \((A^{ss})^{ss}(I) = \{x \in X : U_x \cap A^{ss} \notin I, \text{ for each } U_x \in SN(x)\}\). Thus, we have \((A^{ss})^{ss}(I) \subseteq \{x \in X : U_x \cap A \notin I, \text{ for each } U_x \in SN(x)\} = A^{ss}(I)\).

(iv) It follows from (ii) that \(A^{ss} \cup B^{ss} \subseteq (A \cup B)^{ss}\) ................. (1)

Now, assume that \(x \in (A \cup B)^{ss}\), implies for every \(U \in SN(x)\), \(U_x \cap (A \cup B) \notin I\), implies for every \(U \in SN(x)\), \((U_x \cap A) \cup (U_x \cap B) \notin I\).

Thus, \(x \in A^{ss} \cup B^{ss}\). Therefore \((A \cup B)^{ss} \subseteq A^{ss} \cup B^{ss}\) ................. (2)

(1) and (2) complete the proof.

(v) follows from (ii) and the fact that \(A \cap B \subseteq A\).

(vi) follows directly from the definition of semi local function and the fact that \((U_x \cap A) \setminus (U_x \cap B) = U_x \cap (A \setminus B)\).

(vii) clear, since \(A^{ss}(I)\) is semi closed.

(viii) The technique of the proof is similar to that of statement (iv).

(ix)

\[
U \cap A^{ss} = U \cap \{x \in X : U \cap A \notin I, \text{ for each } U \in SN(x)\} \\
= U \cap \{x \in X : U \cap (U \cap A) \notin I, \text{ for each } U \in SN(x)\} \\
= U \cap (U \cap A)^{ss} \subseteq (U \cap A)^{ss}
\]

**Corollary 2.2.** If \((X, T, I)\) is a space, and \(A \subseteq X\), then the following hold:

(i) \((A^{ss})^{ss} \subseteq A^{ss} = s-Cl(A^{ss}) \subseteq s-Cl(A)\),

(ii) \((A^{ss})^{ss} \subseteq s-Cl(A^{ss}) \subseteq Cl(A)\),

(iii) \(s-Cl(A^{ss}) = Cl(A^{ss}) \text{ if } A^{ss} \text{ is dense}\),

(iv) \(A^{ss} \subseteq A^{*} \subseteq Cl(A)\).

**Proof.** (i) The result follows immediately from Theorem 2.1 (iii) and (vii).

(ii) and (iii) are obvious.

(iv) follows from Remark 2.1 (i) and the fact that \(A^{*}(I)\) is a closed subset of \(Cl(A)\).

**Theorem 2.2.** Let \((X, T, I)\) be a space and \(A \subseteq X\), then:

(i) \((X \setminus E)^{ss} = X^{ss}\), if \(E \in I\).
(ii) \([X\setminus(A\setminus E)]^* = [(X\setminus A) \cup E]^*\), if \(E \in I\).

Proof. (i) Let \(x \in (X\setminus E)^*\), then for every \(U_x \in SN(x), U_x \cap (X\setminus E) \notin I\), implies, for every \(U_x \in SN(x), (U_x \cap X) \setminus (U_x \cap E) \notin I\), implies, for every \(U_x \in SN(x), U_x \cap X \notin I\). Hence \(x \in X^*\), and therefore \((X\setminus E)^* \subseteq X^*\).

Also, let \(x \in X^*\), then \(U_x \cap X \notin I\), for every \(U_x \in SN(x)\), implies, \(U_x \cap (X\setminus E) \notin I\), for every \(U_x \in SN(x)\). Thus, \(x \in (X\setminus E)^*\) and this complete the proof.

(ii) Follows directly by using Theorem 2.1 (viii).

The following theorem gives some properties for the operator \(Cl^*\).

**Theorem 2.2.** For arbitrary subsets \(A\) and \(B\) of \(X\), we have:

(i) \(A \subseteq B\) implies \(Cl^*(A) \subseteq Cl^*(B)\),

(ii) \(Cl^*(A \cup B) = Cl^*(A) \cup Cl^*(B)\),

(iii) \(Cl^*(A \cap B) \subseteq Cl^*(A) \cap Cl^*(B)\),

(iv) \(Cl^*(Cl^*(A)) \subseteq Cl^*(A)\).

Proof. (i), (ii): Follows from the definition of \(Cl^*\) and Theorem 2.1(i),

(iv).

(iii) Evident by using (i).

(iv) \(Cl^*(Cl^*(A)) = Cl^*(A) \cup (Cl^*(A))^* = A \cup A^* \cup (A \cup A^*)^*\),

by using Theorem 2.1 (iv) we have: \(Cl^*(Cl^*(A)) \subseteq A \cup A^* = Cl^*(A)\),

since \((A^*)^* \subseteq A^*\).

**Theorem 2.3.** For any \(A, B \subseteq (X, T, I)\), we have

\[ s-Cl(A^*) \cup s-Cl(B^*) \subseteq s-Cl(A^* \cup B^*). \]

Proof. Follows directly from the fact that \(s-Cl(A) = A \cup \text{int}(Cl(A))[1].\)

**Theorem 2.4.** Let \((X, T)\) be a space with \(I\) and \(J\) are two ideals on \(X\), and let \(A\) be a subset of \(X\), then: \(A^*(J) \subseteq A^*(I)\) if \(I \subseteq J\).

Proof.

\[ A^*(J) = \{ x \in X : U_x \cap A \notin J, \text{ for every } U_x \in SN(x) \} \]

\[ \subseteq \{ x \in X : U_x \cap A \notin I, \text{ for every } U_x \in SN(x) \} = A^*(I). \]

**Theorem 2.5.** If \(I\) and \(J\) are two ideals on \((X, T)\) such that \(I \subseteq J\), then \(T^*(I) \subseteq T^*(J)\).
Proof. Follows from Theorem 2.4.

Theorem 2.6. If \((X,T,I)\) is a space and \(A \subseteq X\), then \(A^*\setminus(A^*)^s \subseteq (A\setminus A^s)^s\)

Proof. Assume that \(x \in A^*\setminus(A^*)^s\), then \(x \in A^s\), i.e., for each \(U \in SN(x)\), \(U \cap A \not\subseteq I\). Thus, for each \(U \in SN(x)\), \(U \cap (A\setminus A^s) \not\subseteq I\), which implies \(x \in (A\setminus A^s)^s\) and this complete the proof.

Theorem 2.7. Let \((X,T)\) be a space with \(I\) and \(J\) are two ideals on \(X\) and \(A \subseteq X\). Then: \(A^s(I \cap J) = A^s(I) \cup A^s(J)\) where \(I \cap J\) is an ideal on \(X\).

Proof.

\[
A^s(I \cap J) = \{x \in X : U_x \cap A \not\subseteq I \cap J, \text{ for each } U_x \in SN(x)\} = \{x \in X : U_x \cap A \not\subseteq I, \text{ or } U_x \cap A \not\subseteq J, \text{ for each } U_x \in SN(x)\} = A^s(I) \cup A^s(J).
\]

Theorem 2.8. Let \((X,T)\) be a space with \(I\) and \(J\) ideals on \(X\) and \(A \subseteq X\). Then \(A^s(IvJ,T) = A^s(I,T^s(J)) \cap A^s(J,T^s(I))\) where \(IvJ = \{E \cup H : E \in I \text{ and } H \in J\}\) is an ideal on \(X\).

Proof. Suppose \(x \not\in A^s(IvJ,T)\), then there exists \(U \in SN(x)\) such that \(U \cap A \in IvJ\). Let \(E \in I\) and \(H \in J\) such that \(U \cap A = E \cup H\), we may assume \(E \cap H = \emptyset\). Thus we have, \((U \cap A) \setminus E = H\) and \((U \cap A) \setminus H = E\), implies, \((U \setminus E) \cap A = H \in J\) and \((U \setminus H) \cap A = E \in I\), implies, \(x \not\in A^s(J,T^s(I))\) or \(x \not\in A^s(I,T^s(J))\) \((x\) could be in \(E\) or \(H\), but not both\).

We have shown

\[
A^s(J,T^s(I)) \cap A^s(I,T^s(J)) \subseteq A^s(IvJ,T) \tag{1}
\]

Now, we have to prove that, \(A^s(IvJ,T) \subseteq A^s(I,T^s(J))\). Let \(x \not\in A^s(I,T^s(J))\). This implies that there exists \(U \in SN(x)\) and \(H \in J\) such that \((U \setminus H) \cap A \in I\). We may assume, because of the heredity of \(J\), that \(H \subseteq A\). Now define \(E = (U \setminus H) \cap A\), and we have \(U \cap A = E \cup H \in IvJ\) implies \(x \not\in A^s(IvJ,T)\).

We have shown, \(A^s(IvJ,T) \subseteq A^s(I,T^s(J))\) \(\tag{2}\)

Similarly, we have that \(A^s(IvJ,T) \subseteq A^s(J,T^s(I))\) \(\tag{3}\)

From (2) and (3), we have that

\[
A^s(IvJ,T) \subseteq A^s(I,T^s(J)) \cap A^s(J,T^s(I)) \tag{4}
\]

(1) and (4) establish the result.

Corollary 2.3. Let \((X,T)\) be a space and \(I\) an ideal on \(X\). Then \(A^s(I,T) = A^s(I,T^s)\) and hence \(T^s = (T^s)^s\).
Definition 2.2. A subset \( W \subseteq (X, T, I) \) is called \( T^{*s} \)-closed iff \( W^{*s} \subseteq W \).

The proof of the following theorem is straightforward, so we omit it.

Theorem 2.9. The following statements are equivalent for a subset \( W \subseteq (X, T, I) \).

(i) \( W \in T^{*s} \),
(ii) \( (X \setminus W) \) is \( T^{*s} \)-closed,
(iii) \( (X \setminus W)^{*s} \subseteq (X \setminus W) \),
(iv) \( W \subseteq X \setminus (X \setminus W)^{*s} \).

3. The set operator \( \Psi^s(I, T) \)

Definition 3.1. Let \( (X, T, I) \) be a space, a set operator \( \Psi^s(I, T) : P(X) \rightarrow \mathcal{P}(X) \) defines as follows: for every \( A \subseteq X \), \( \Psi^s(I, T)(A) = \{ x : \text{there exists a } U \in SN(x) \text{ such that } U \setminus A \in I \} \), equivalently, \( \Psi^s(I, T)(A) = X \setminus (X \setminus A)^{*s} \) i.e. the operator \( \Psi^s(I, T) \) is a natural complement to the operator \((\cdot)^{*s}\). We denote \( \Psi^s(I, T) \) simply by \( \Psi^s \) when no ambiguity is present.

Corollary 3.1. For \( A \subseteq (X, T, I) \), we have:

(i) If \( I = \{ \emptyset \} \), then \( \Psi^s(A) = s\text{-int}(A) \).
(ii) If \( I = P(X) \), then \( \Psi^s(A) = X \).

Proof. By taking the complementation of corollary 2.1.

Remark 3.1. One can deduce that \( \Psi(A) \subseteq \Psi^s(A) \), for every \( A \subseteq (X, T, I) \) and the converse is not true as shown by the following example.

Example 3.1. If \( X = \{ a, b, c \} \), \( T = \{ X, \emptyset, \{ a \}, \{ b \}, \{ a, b \} \} \) and \( I = \{ \emptyset, \{ c \} \} \). It is clear that if \( A = \{ a \} \), then \( \Psi(A) = \{ a \} \), but \( \Psi^s(A) = \{ a, c \} \).

The following theorem gives many basic and useful facts for the operator \( \Psi^s \).

Theorem 3.1. Let \( (X, T, I) \) be a space and \( A, B \subseteq P(X) \), then

(i) if \( A \subseteq B \), then \( \Psi^s(A) \subseteq \Psi^s(B) \),
(ii) if \( A, B \in P(X) \), then \( \Psi^s(A \cap B) \subseteq \Psi^s(A) \cap \Psi^s(B) \),
(iii) if \( U \in T^{*s}(I) \), then \( U \subseteq \Psi^s(U) \),
(iv) if \( A \in P(X) \), then \( \Psi^s(A) \subseteq \Psi^s(\Psi^s(A)) \),
(v) if \( A \in P(X) \), then \( \Psi^s(A) = \Psi^s(\Psi^s(A)) \) iff \( (X \setminus A)^{*s} = ((X \setminus A)^{*s})^{*s} \).
(vi) if \( A \in I \), then \( \Psi^s(A) = X \setminus X^{**} \),

(vii) if \( A \in P(X) \), \( E \in I \), then \( \Psi^s(A \setminus E) = \Psi^s(A) \),

(viii) if \( A \in P(X) \), \( E \in I \), then \( \Psi^s(A \cup E) = \Psi^s(A) \),

(ix) if \( (A \setminus B) \cup (B \setminus A) \in I \), then \( \Psi^s(A) = \Psi^s(B) \).

**Proof.**

(i) Since \( A \subseteq B \), then \( (X \setminus B)^{**} \subseteq (X \setminus A)^{**} \).

Thus, \( X \setminus (X \setminus A)^{**} \subseteq X \setminus (X \setminus B)^{**} \), from the definition of the operator \( \Psi^s \), we get the result.

(ii) Follows from the fact that \( A \cap B \subseteq A \) and by (i).

(iii) If \( U \in T^s(I) \), then \( X \setminus U \) is \( T^s \)-closed, which implies \( (X \setminus U)^{**} \subseteq (X \setminus U) \), and hence \( U \subseteq X \setminus (X \setminus U)^{**} = \Psi^s(U) \).

(iv) Since \( \Psi^s(A) \) is semi-open, then by using (iii) we have \( \Psi^s(A) \subseteq \Psi^s(\Psi^s(A)) \).

(v) Since \( \Psi^s(A) = X \setminus (X \setminus A)^{**} \). Then,

\[
\Psi^s(\Psi^s(A)) = \Psi^s(X \setminus (X \setminus A)^{**})
= X \setminus [X \setminus (X \setminus A)^{**}]^{**}
= X \setminus [(X \setminus A)^{**}]^{**}, \text{ from hypothesis}
= X \setminus (X \setminus A)^{**} = \Psi^s(A).
\]

(vi) Since \( \Psi^s(A) = X \setminus (X \setminus A)^{**} \). Then by using Theorem 2.2(i) we have, \( \Psi^s(A) = X \setminus X^{**} \) for \( A \in I \).

(vii)

\[
\Psi^s(A \setminus E) = X \setminus (X \setminus (A \setminus E))^{**}
= X \setminus ((X \setminus A) \cup E)^{**}, \text{ from Theorem 2.1(viii)}
= X \setminus (X \setminus A)^{**} = \Psi^s(A).
\]

(viii)

\[
\Psi^s(A \cup E) = X \setminus (X \setminus (A \cup E))^{**}
= X \setminus ((X \setminus A) \cup E)^{**}, \text{ by using Theorem 2.1(viii)}
= X \setminus (X \setminus A)^{**} = \Psi^s(A).
\]

(ix) Assume that \( (A \setminus B) \cup (B \setminus A) \in I \), and let \( A \setminus B = E \), \( B \setminus A = H \). Observe that \( E, H \in I \) by heredity. Also, observe that \( B = (A \setminus E) \cup H \).

Thus,

\[
\Psi^s(A) = \Psi^s(A \setminus E) \text{ (from (vii))}
= \Psi^s[(A \setminus E) \cup H] \text{ (from (viii))}
= \Psi^s(B).
\]
Corollary 3.1. If $A \subseteq (X, \mathcal{T}, I)$ and $E \in I$, then: $\Psi^s(A \setminus E) = \Psi^s(A \cup E) = \Psi^s(A)$.

Proof. Follows from Theorem (3.1) (vii) and (viii).

Newcomb [10] defined $A = B \ [\text{mod } I]$ if $(A \setminus B) \cup (B \setminus A) \in I$ and observed that $"= [\text{ mod } I]"$ is an equivalence relation. Let us denote by $A \Delta B$ the "symmetric difference" $(A \setminus B) \cup (B \setminus A)$.

Theorem 3.2. If $A = B \ [\text{mod } I]$ then $\Psi^s(A) = \Psi^s(B)$ for every $A, B \in P(X)$.

Proof. It is clear from the definition of $A = B \ [\text{mod } I]$ and Theorem 3.1 (ix).

4. Semi compatibility of $\mathcal{T}$ with $I$

Definition 4.1. Given a space $(X, \mathcal{T}, I)$, $I$ is said to be semi-compatible with respect to $\mathcal{T}$, denoted by $I \sim \mathcal{T}$, if the following holds: if for every $A \subseteq X$ and for every $x \in A$, there exists $U \in SN(x)$ such that $U \cap A \in I$, then $A \in I$.

Remark 4.1. One can easily show that semi compatibility implies compatibility i.e. $I \sim \mathcal{T} \Rightarrow I \sim \mathcal{T}$.

The following example show the existence of this new type of compatibility.

Example 4.1. Let $X = \{a, b, c, d\}$, $\mathcal{T} = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. It is clear that $I \sim \mathcal{T}$.

The following theorem gives some characterizations of semi compatibility.

Theorem 4.1. If $(X, \mathcal{T}, I)$ is a space. Then the following statements are equivalent.

(i) $I \sim \mathcal{T}$,

(ii) If $A$ has a cover of semi open sets each of whose intersection with $A$ is in $I$, then $A$ is in $I$,

(iii) For every, $A \subseteq X$, $A \cap A^{**} = \emptyset$ implies $A \in I$,

(iv) For every $A \subseteq X$, $A \setminus A^{**} \in I$,

(v) For every $\mathcal{T}^{**}$-closed subset $A$, $A \setminus A^{**} \in I$,
(vi) For every $A \subseteq X$, if $A$ contains no nonempty subset $B$ with $B \subseteq B^*$, then $A \in I$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv): Let $A \subseteq X$, then by (iii) $(A \setminus A^*) \cap (A \setminus A^*)^* = \emptyset$ which implies $(A \setminus A^*) \in I$.

(iv) $\Rightarrow$ (v): Straightforward.

(v) $\Rightarrow$ (i): Let $A \subseteq X$, and for every $x \in A$, there exists $U \in SN(x)$ such that $U \cap A \in I$. Then $A \cap A^* = \emptyset$ and since $(A \cup A^*)$ is $T^*$-closed, then by (v), we have $(A \cup A^*) \setminus (A \cup A^*)^* \in I$, and

$$(A \cup A^*) \setminus (A \cup A^*)^* = (A \cup A^*) \setminus (A \cup (A)^*^*)$$

$$= (A \cup A^*) \setminus A^* \text{ (for } (A)^*^* \subseteq A^*)$$

$$= A. \text{ Therefore } A \in I.$$

(iv) $\Rightarrow$ (vi): Let $A \subseteq X$ and let $A$ has no nonempty subset $B$ with $B \subseteq B^*$. Since $A \setminus A^* \in I$, (by iv), $A \cap A^* \subseteq (A \cap A^*)^*$, and hence $A \cap A^* = \emptyset$. Therefore $A = A \setminus A^*$ and $A \in I$.

(vi) $\Rightarrow$ (iv): Let $A \subseteq X$. Since $(A \setminus A^*) \cap (A \setminus A^*)^* = \emptyset$, then $A \setminus A^*$ contains no nonempty subset $B$ with $B \subseteq B^*$. Hence $A \setminus A^* \in I$.

**Theorem 4.2.** Let $(X, T, I)$ be a space. Then $I \approx T$ iff $\psi^*(A) \setminus A \in I$ for every $A \subseteq X$.

Proof. (Necessity). Assume that $I \approx T$ and let $A \subseteq X$. Observe that $x \in \psi^*(A) \setminus A$ iff $x \not\in A$ and $x \not\in (X \setminus A)^*$ iff $x \not\in A$ and there exists a $U_x \in SN(x)$ such that $U_x \setminus A \in I$ iff there exists a $U_x \in SN(x)$ such that $x \in U_x \setminus A \in I$. Now, for each $x \in \psi^*(A) \setminus A$ and $U_x \in SN(x)$, $U_x \cap (\psi^*(A) \setminus A) \in I$ by heredity so that $\psi^*(A) \setminus A \in I$ by assumption that $I \approx T$.

(Sufficiency). Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in SN(x)$ such that $U_x \cap A \in I$. Observe that $\psi^*(X \setminus A) \setminus (X \setminus A) = \{x: \text{ there exists } U_x \in SN(x) \text{ such that } x \in U_x \cap A \in I\}$. Thus we have $A \subseteq \psi^*(X \setminus A) \setminus (X \setminus A) \in I$ and hence $A \in I$ by the heredity of $I$.

**Theorem 4.3.** Let $(X, T, I)$ be a space with $I \approx T$. Then $\psi^*(\psi^*(A)) = \psi^*(A)$ for every $A \subseteq X$.

Proof. From Theorem 3.1 (iii), we have, $\psi^*(A) \subseteq \psi^*(\psi^*(A))$. Since $I \approx T$, $\psi^*(A) = A \cup E$ for some $E \in I$ (Theorem 4.2) and hence $\psi^*(\psi^*(A)) = \psi^*(A \cup E) = \psi^*(A)$ by Theorem 3.1 (vii).
Theorem 4.4. Let \((X,T,I)\) be a space with \(I \sim T\). If \(U,V \in SO(X)\) and \(\Psi^s(U) = \Psi^s(V)\), then \(U = V [\text{mod } I]\).

Proof. \(U \subseteq \Psi^s(U)\) (by Theorem 3.1 (iii)) implies \(U \setminus V \subseteq \Psi^s(U) \setminus V = \Psi^s(V) \setminus V \in I\) (Theorem 4.2).

Similarly \(V \setminus U \in I\). Therefore, \((U \setminus V) \cup (V \setminus U) \in I\) by additivity. Thus \(U = V [\text{mod } I]\).

Theorem 4.5. Let \((X,T,I)\) be a space such that \(I \sim T\). Then \((A^s)^* = A^s\) for every \(A \subseteq X\).

Proof. Obvious.

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References


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