GROUP ACTIONS ON DONALDSON INVARIANTS

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Dedicated to Professor Younki Chae on his 60th birthday

1. Introduction

Let $M$ be a smooth, closed, simply connected 4-manifold. Let a group $G$ acts on $M$ as isometries. Suppose that $E \xrightarrow{\pi} M$ is on $SU(2)$ vector bundle and the $G$-action on $M$ lifts to the bundle $E$ such that $\pi$ is a $G$-equivariant map. In [8] Finashin, Kreck and Viro studied on anad $\alpha$-involutions on the Dolgachev surfaces. They got non-diffeomorphic but homeomorphic knottings in the 4-sphere. In [9] Fintushel and Stern studied on the equivariant moduli space of self-dual connection. They have infinite cobordism group of homology 3-spheres. In [2] and [3] we studied on finite group actions on the moduli space of self-dual connections. We have some cohomology obstruction classes to get a smooth $G$-moduli space. In [6] Donaldson gave a comment on anti-holomorphic involution on $K3$-space.

The $G$-bundle $E \xrightarrow{\pi} M$ induces an $SU(2)$ orbit bundle $E_1 \xrightarrow{\pi} N$ where $N$ is the $G$-orbit space of $M$. The moduli space of gauge equivalence classes of anti-self-dual connections of the $G$-bundle $E$ is also a $G$-space. In this paper we would like to study the relations between the $G$-moduli space and the moduli space for the orbit bundle $E_1$. Roughly speaking, Donaldson invariants are just the cohomology pairings on the moduli spaces. We would like to study the Donaldson invariants on the $G$-moduli space and the moduli space obtained from the orbit bundle and their relations.

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321
In section 2 we introduce simple examples of algebraic surfaces. Dolgachev surface, fibersum and logarithmic transformation.

In section 3 we introduce Finashin, Kreck and Viro’s results on anti-holomorphic involution on Dogachev surfaces and anti-holomorphic involution $K3$-surface. We study the topology on the orbit spaces. In section 4 we study equivariant geometry on vector bundles, the space of connections and the moduli space of anti-self-dual connections. In section 5 we study involution on $K3$ surface. The involution induces an involution on the moduli space of anti-self-dual connections. We compute the dimension of the invariant moduli space. We show that the moduli space on the orbit bundle can be identified one of the components of the invariant moduli space. In section 6 we show that Donaldson $\mu$-map is functorial to the projection from given space onto the orbit space. We introduce the determinant line bundle which induces the $\mu$-map. We compute a curvature form of the line bundle as Lie algebra valued 2-form. In section 7 we introduce the results of Friedman and Morgan about homeomorphic but not diffeomorphic examples. Finally for an anti-holomorphic involution on a $K3$ surface, we show that the Donaldson invariant on the moduli space obtained from the orbit bundle can be computed by the fixed point set of the $G$-moduli space, and that the anti-holomorphic involution preserves the Donaldson invariants.

2. Algebraic Surfaces

The simple examples of a smooth, closed, simply connected algebraic surfaces are $\mathbb{C}P^2$, $S^2 \times S^2$ an the $K3$ surface. They are zero sets in $\mathbb{C}P^3$ of polynomials of degree 1, 2 and 4 respectively. An elliptic surface is a compact complex surface $S$ which is the total space of a holomorphic fibration $S \rightarrow C$ onto a Riemann surface such that the generic fibres of $\pi$ are complex tori. As an example we would like to introduce the simplest elliptic surface. We start with a generic pencil of cubic curves on $\mathbb{C}P^2$. Let $f_1$ and $f_2$ be a pair of homogeneous cubic polynomials on $\mathbb{C}^3$ in general position. For each point $x = [x_0, x_1] \in \mathbb{C}P^1$ in the homogeneous coordinates, let $Z_x$ be the zero set of the polynomial $x_0 f_1 + x_1 f_2$ in $\mathbb{C}P^2$. Then $Z_x$ is a smoothly embedded torus in $\mathbb{C}P^2$ except for finitely many values of $x \in \mathbb{C}P^1$. Let $Z$ be the intersection of the zero sets of $f_1$ and $f_2$. Since $f_1$ and $f_2$ are cubic polynomials in general position, the set $Z$ is the set of 9 distinct points. The set $Z$ is a subset of the zero set $Z_x$ for all $x \in \mathbb{C}P^1$. For each point $p \in \mathbb{C}P^2 - Z$ there is a unique point
Let \( x \in \mathbb{CP}^1 \) such that \( p \in Z_x \). If we blow up the 9 points in \( Z \), the zero sets \( Z_x \) are disjoint. We have an elliptic fibration \( \mathbb{CP}^2 \# 9 \mathbb{CP}^2 \to \mathbb{CP}^1 \) given by \( \pi(Z_x) = x \). Let \( D_1 = \mathbb{CP}^2 \# 9 \mathbb{CP}^2 \).

Let \( N \subset D_1 \) be the inverse image of \( \pi \) of a small open 2-disc in \( \mathbb{CP}^1 \). Let \( \phi : \partial N \to \partial N \) be a fiber-preserving, orientation reversing self-diffeomorphism where \( \partial N \) is the boundary of \( N \) which is diffeomorphic to \( T^2 \times S^1 \). If we glue \( D_1 \setminus N \) and \( D_1 \setminus N \) along their boundaries via the map \( \phi \). This is called a fiber sum which is not holomorphic processes in general. Since \( D_1 \) has Euler characteristic \( \chi(D_1) = 12 \) and signature \( \sigma(D_1) = -8 \), we have \( \chi(D_2) = 24 \) and \( \sigma(D_2) = -16 \). In fact the fiber sum \( D_2 \) is diffeomorphic to the \( K3 \)-surface. The fiber sum turns out to be independent of all choices.

We introduce another operation on an elliptic surface \( S \to C \) which is called logarithmic transformation. In the smooth category let \( N \) be the inverse image of a small neighbourhood of a generic point in \( C \) as before. We delete \( N \) and glue \( T^2 \times D^2 \) back in by some diffeomorphism \( \phi : T^2 \times S^1 \to \partial N \). The multiplicity is defined to be the absolute value of the winding number of \( \pi \circ \phi \): point \( \times S^1 \to \pi(\partial N) \cong S^1 \). In the holomorphic category for a generic point \( p \in C \) the fiber \( S_p \) is a complex torus. Let \( \Delta_t \) be a small coordinate disc about \( p \) in \( C \). Over \( \Delta_t \) our fibration is isomorphic to a fibration \( X = C \times \Delta/L \) where \( L = (Z + Z\omega(t)) \) and \( \omega \) holomorphic on \( \Delta \) and \( \text{Im} \omega(t) \times 0 \). Let \( t = s^m \). we take a \( k \in N \) with \( (k,m) = 1 \) and form the smooth quotient \( X' = C \times \Delta_s/(Z + Z\omega(s^m)) \) by the cyclic group of order \( m \) generated by \( (z,s) \to (z + k/m, e^{2\pi i /m}s) \). We define \( X' \to \Delta_t \) by \( (z,s) \to s^m = t \). We have an elliptic fibration \( X' \to \Delta_t \) which has only one singular fiber \( X_0' \) at 0, of multiplicity \( m \). The map given by \( (z,s) \to (z - k/2\pi i \log s, s^m) \) induces a biholomorphic fiber preserving map \( h : X' \setminus x_0' \to X \setminus x_0 \). If we identify \( X' \setminus X_0' \) with the \( S|_{\Delta \setminus \{0\}} \to \Delta \setminus \{0\} \subset C \) via \( h \), then the union \( (S|_{\Delta} \setminus sp) \cup X_0' \) becomes an elliptic fibration \( S' \to C \) with the two properties \( S'|_{sp} \) biholomorphic to \( S|_{sp} \) and \( sp' \) has multiplicity \( m \). Some \( K3 \)-surfaces admit elliptic fibrations.

Let \( p \) and \( q \) be relatively prime integers and \( X(p,q) \) denote the manifold obtained from a \( K3 \)-surface \( X \) by logarithmic transformations of multiplicities \( p \) and \( q \). If \( p \) and \( q \) are both odd, then \( X(p,q) \) admits a spin structure. Applying Freedman classification theorem for simply connected, closed topological 4-manifolds, we have

**Theorem 2.1.** If \( X(p,q) \) is obtained from a \( K3 \)-surface \( X \) by logarithmic
transformation of multiplicities $p$ and $q$.

1. $X(p, q)$ is homeomorphic to the $K3$-surface $X$ if $p$ and $q$ are odd.
2. $X(p, q)$ is homeomorphic to $3\mathbb{CP}^2 \# 19\mathbb{CP}^2$ otherwise.

3. Involution on Algebraic Surface

Let $c$ be the standard antiholomorphic involution on $\mathbb{CP}^2$. The fixed point set of $c$ is the standard embedding of $\mathbb{RP}^2$ into $S^4$ with normal Euler number $-2$ in $\mathbb{CP}^2/c \cong S^4$. There are antiholomorphic involution con the Dolgachev surface $D_1(2, 2k + 1)$ with fixed point set $F(k) = \#10\mathbb{RP}^2$ with normal Euler number 16 is obtained by taking the connected sum $(S^4, \mathbb{RP}^2) \#9(-S^4, \mathbb{RP}^2) = (S^4, F(k))$

Theorem 3.1. (Finashin, Kreck, Viro) (1) The knottings $(S^4, F(k_1))$ and $(S^4, F(k_2))$ are not diffeomorphic for $k_1 \neq k_2$.

2. $F(k) = \#10(\mathbb{RP}^2)$.

3. $\pi_1(S^4 \setminus F(k)) = \mathbb{Z}_2$.

4. The normal Euler number with local coefficients of $F(k)$ in $S^4$ is 16.

5. The knottings $(S^4, F(k))$ are homeomorphic.

More generally let us start with a complex algebraic surface $X$ defined over the real numbers. So there is an antiholomorphic involution $\sigma : X \to X$ with fixed point set a real form $X_R$ of $X$ which is a real algebraic surface. Let $Y$ be the quotient space $X \setminus \sigma$. Since the fixed point set has real codimension 2, $Y$ is a smooth manifold.

Lemma 3.2. (1) If $X$ is simply connected, then the quotient $Y$ is also simply connected since the nontrivial loop in $Y$ can be lifted to be a nontrivial loop in $X$.

2. $b^*(Y) = p_\sigma(X)$.

3. $2\chi(Y) = \chi(X) + \chi(X_R)$.

4. The manifold $X$ can be recovered by the double cover of $Y$, branched over the fixed point set $X_R$.

Theorem 3.3. (Donaldson) Let $(S, \sigma)$ be a $K3$ surface with antiholomorphic involution $\sigma$. Then $S/\sigma$ is one of standard rational surfaces $S^2 \times S^2$ or $\mathbb{CP}^2 \# n\mathbb{CP}^2$.

Proof. Yau's solution of the Calabi conjecture gives a $\sigma$-invariant Kähler metric on $S$, compatible with a family of complex structures. With respect to one such complex structure, $\sigma$, the map $\sigma$ is a holomorphic involution
In fact this complex structure is orthogonal to the original one present in our explicit complex description of $S$. Then $J$ induces a complex structure on the quotient space $Y = S/\sigma$, such that the projection map is a holomorphic branched cover. But it is a simple fact from the complex surface theory that a $K3$ surface is a branched cover of a surface $Y$ than $Y$ is a rational surface.

**Remark 3.4.** By the argument of Finashin, Kreck and Viro, the quotient of a logarithmic transform $D_2(p, q)$ by an antiholomorphic involution is again diffeomorphic to one of these standard manifolds. We get knotted complex curves in this manifold, that is, embedded surfaces homologous to a complex curve of the same genus, but not isotopic to a complex curve.

Let $\sigma$ be the standard antiholomorphic involution on a $K3$-surface $X$. Then the fixed point set $X_R$ is a real algebraic variety of real dimension 2. Let the Kähler metric be $h = \sum h_{i\bar{j}}dz^i \otimes d\bar{z}^j$ in the theorems. The Kähler form is $\frac{i}{2} \sum h_{i\bar{j}}dz^i \wedge d\bar{z}^j$. Since $\pi : X \to Y, Y = X/\sigma$ is a holomorphic branched cover with branch set $X_R$.

If $p \in X \setminus X_R$, then clearly there is a neighbourhood $U$ of $p$ such that $\pi|_U$ is holomorphic ($b_i$). That is $\pi$ is double cover on the subset $X \setminus X_R$, $\pi :$ local diffeomorphism and $\pi^*h = h$ on $Y \setminus X_R$.

If $p \in X_R$, then there is a neighbourhood $U$ of $p$ in $X$ such that $U \simeq U_1 \times U_2$ where $\sigma$ is trivial on $U_1, U_2 \simeq D^2$ and $\sigma$ is the reflection with respect to the origin. Thus

$$U/\sigma \simeq (U_1 \times U_2)/\sigma \simeq U_1 \times (U_2/\sigma) \simeq U_1 \times U_2.$$  

Here we can choose $\sigma(U) = U$ by taking $U = V \cap \sigma(V)$ where $V$ is a neighbourhood of $p$.

Let $G = \{1, \sigma\}$ be the group of order 2. The intersection form of $X$ is $-2E_8 \oplus 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and is positive on $H^2_+(X) \cong \mathbb{Z}^3$ and is negative on $H^2(X) \cong \mathbb{Z}^{19}$. By Lefschetz-fixed point theorem.

**Theorem 3.5.**

$$\text{sign } X \setminus G = \frac{1}{|G|} \sum_{\sigma \in G} (\text{trg}|_{H^2_+} - \text{trg}|_{H^2_-}) = \frac{1}{2}[(3 - 19) + (\text{tr}\sigma|_{H^+} - \text{tr}\sigma|_{H^-})].$$

By $G$-signature theorem for the involution $\sigma : X \to X$.

$$\text{sign } (\sigma, X) = \text{tr}\sigma|_{H^2_+(X)} - \text{tr}\sigma|_{H^2_-(X)} = \text{sign}(X)^2 = \text{sign}(X_R)^2$$
where $X_R$ is a real 2-dimensional surface in $X$. $X_R^2$ is a zero dimensional submanifold of $X$ consisting of finitely many isolated points with sign. Thus we have

**Theorem 3.6.** $\text{sign} (Y) = \frac{1}{2}[-16 + \text{sign}(X_R)^2]$. In particular, if $Y = S^2 \times S^2$, then $\text{sign} (Y) = 0$ and $\text{sign}(X_R)^2 = 16$. If $Y = \mathbb{C}P^2 \# n\mathbb{C}P^2$, then $\text{sign} (Y) = 1 - n = \frac{1}{2}[-16 + \text{sign}(X_R)^2]$ and $\text{sign}(X_R)^2 = 2(1 - n) + 16$.

**4. The Equivariant Geometry on Bundles**

Let $G$ be a finite abelian group. Let $G$ act on a smooth, closed, simply connected 4-manifold $M$ and lift to an $SU(2)$ vector bundle $\pi : E \to M$ on which $\pi$ is a $G$-map.

Choose Riemannian metrics on $M$ and $E$ on which $G$ acts as isometries. A Riemannian connection is a linear map $\nabla : \Omega^0(E) \to \Omega^1(E)$ which satisfies

$$
\nabla(f \sigma) = df \otimes \sigma + f\nabla \sigma
$$

$$
d < \sigma, \tau > = < \nabla \sigma, \tau > + < \sigma, \Delta \tau >
$$

where $f \in C^\infty(M)$ and $\sigma, \tau \in \Omega^0(E)$ and $<,>$ is the given Riemannian metric on $E$. There are Riemannian connections on $E$. Let $C$ be the space of all Riemannian connections on $E$. In fact $C$ is an affine space as a model space $\Omega^1(adP)$ where $adP$ is the associated Lie algebra bundle of $E$. For $\nabla \in C$ let $F^\nabla \in \Omega^2(adP)$ be the curvature of $\nabla$. Let $G$ be the group of gauge transformations of $E$ which are the sections of the associated Lie group bundle by the adjoint action of $SU(2)$ on itself. The group $G$ acts on by $g(\nabla) = g \circ \nabla \circ g^{-1}$.

Let $B = C/G$ be the orbit space and $\pi : C \to B$ be the projection. Let $*$ be the Hodge star operator on the oriented Riemannian 4-manifold $M$. A connection $\nabla \in C$ is said anti-self-dual if $*F^\nabla + F^\nabla = 0$.

**Lemma 4.1.** For any $\nabla \in C$ and any $h \in G$, we have $h(\nabla) \in C$.

**Proof.** Let $v \in TM$, $\sigma, \sigma_1, \sigma_2 \in \Omega^0(E)$ and $f \in C^\infty(M)$. $h(\nabla)(f \sigma) = h [\nabla h^{-1} \circ (f \sigma) \circ h] = h[\nabla f \circ (h^{-1} \circ \sigma \circ h)] = h(df \otimes h^{-1} \circ \sigma \circ h) + h(f \nabla h^{-1} \circ \sigma \circ h) = df \otimes \sigma + f[h(\nabla) \sigma]$.

Compatibility of the connection $h(\nabla)$ with the Riemannian structures:

$$
< h(\nabla)v_1, \sigma_1 >_{E_x} + < \sigma_1, h(\nabla)v_2 >_{E_x} = < h(\nabla h^{-1} \circ \sigma \circ h), \sigma_2 >_{E_x} + < \sigma_1, h(\nabla h^{-1} \circ \sigma \circ h) >_{E_x}
$$
Group Actions on Donaldson Invariants

\[ \langle \nabla_{h^{-1}v} h^{-1}\sigma h, h^{-1}\sigma_2 \rangle_{E_{h^{-1}(x)}} + \langle h^1 \sigma, \nabla_{h^{-1}v}(h^{-1}\sigma h) \rangle_{E_{h^{-1}(x)}} \]

by \( h \) is isometry on \( E \)

\[ \langle \nabla_v h^{-1}\sigma h, h^{-1}\sigma h \rangle_{E_{x}} + \langle h^{-1}\sigma_1 h, \nabla_v h^{-1}\sigma_2 h \rangle_{E_{x}} \]

by \( h \) is isometry on \( M = v \)

\[ \langle \nabla_v h^{-1}\sigma h, h^{-1}\sigma h \rangle_{E_{x}} = v < \sigma_1, \sigma_2 > \]

by isometries on \( M \) and \( E \).

Let \( \mathcal{A} \) be the subspace of \( \mathcal{C} \) consisting of all anti-self-dual connections on \( E \). The action of \( \mathcal{G} \) on \( \mathcal{C} \) can restrict on \( \mathcal{A} \). The orbit space \( \mathcal{A}/\mathcal{G} \equiv \mathcal{M} \) is called the moduli space of the gauge equivalence class of anti-self-dual connections on \( E \). The action of \( G \) on the bundle \( E \to M \) induces on action of \( G \) on \( \mathcal{C} \). If \( \sigma \in \Omega^0(E) \) and \( h \in G \) then we define \( h(\sigma) = h \circ \sigma \circ h^{-1} \), where \( h^{-1} \) is an action on \( M \) and \( h \) is an action on \( E \) as a bundle map. If \( v \in TM \), and \( \nabla \in \mathcal{C} \), then we define \( h(\nabla)(\sigma) = h(\nabla_{h^{-1}v} h^{-1} \circ \sigma \circ h) \).

Finally we define an action of \( G \) on \( \Omega^k(adP) \) by \( (h\phi)_{v_1...v_k} = h \circ \phi_{h^{-1}v_1,...,h^{-1}v_k} \).

**Lemma 4.1.** Since \( G \) act on \( M \) as isometries, the action of \( G \) on \( \mathcal{C} \) preserves \( \mathcal{A} \). The set of invariant connections of \( E \) denotes \( \mathcal{C}^G = \{ \nabla \in \mathcal{C} | h(\nabla) = \nabla \} \) and the invariant anti-self-dual connections \( \mathcal{A}^G = \mathcal{C}^G \cap \mathcal{A} \). The \( G \)-equivariant gauge group denotes \( \mathcal{G}^G = \{ g \in \mathcal{G} | hg = gh \text{ for all } h \in G \} \). Then \( \mathcal{G}^G \) acts on \( \mathcal{C}^G \) and \( \mathcal{A}^G \). Let \( B^G = \mathcal{C}^G / \mathcal{G}^G \) and \( M^G = \mathcal{A}^G / \mathcal{G}^G \) and \( \Omega^k(adP)^G \) be the \( G \)-invariant subspace of \( \Omega(adP) \). Connection \( \nabla \) on \( E \) induces a connection \( \nabla \) on \( adP \) by for \( \phi \in \Omega^0(adP) \) and for \( \sigma \in \Omega^0(E) \)

\[ (\nabla \phi)(\sigma) = \nabla(\phi \sigma) - \phi(\nabla \sigma). \]

Then we have the following immediate consequences.

**Lemma 4.2.** If \( \nabla \) is a \( G \)-invariant anti-self-dual connection.

(1) \( 0 \to \Omega^0(adP)^G \xrightarrow{\nabla} \Omega^1(adP)^G \xrightarrow{d^G} \Omega^2(adP)^G \to 0 \) is an elliptic complex.

(2) \( F^\nabla \in \Omega^2(adP)^G \).

(3) \( \mathcal{G}^G_{k+1} \) acts on \( \mathcal{C}^G_k \) and \( \mathcal{A}^G_k \) where \( \mathcal{C}^\alpha_k \) is the Sobolev-completion \( \{ \nabla + A | A \in L^2_k(Q^1(adP)^G) \} \) by the \( L^2_k \)-norm.

**Remark 4.3.** If the bundle \( E \to M \) is an \( SO(3) \)-bundle. Let \( \pi : C \to B \) be the projection. Then

\[ B^{G*} = C^{G*} / \mathcal{G}^G = \pi(C^G)^* \]

\[ M^{G*} = A^{G*} / \mathcal{G}^G = \pi(A^G)^*. \]
Proof. Suppose $\nabla$ is a $G$-invariant irreducible connection and $\nabla' = g(\nabla)$. For each $h \in G$, $h(g(\nabla)) = h(\nabla') = g(\nabla) = g(h(\nabla))$ and $g^{-1}h^{-1}gh(\nabla) = \nabla$. $g^{-1}h^{-1}gh$ is an element of the isotropy subgroup $\Gamma^\nabla$ of the gauge transformation group $G$. Since the connection $\nabla$ is irreducible, the gauge transformation $g$ is $G$-invariant.

5. Involution on $K3$ surface

Let an anti-holomorphic involution $\sigma$ on a $K3$-surface $X$ lift to an $SU(2)$ vector bundle $E$ on $X$ on which $\pi$ is $\sigma$-equivariant.

Remark 5.1. Consider the following diagram

$\begin{align*}
E & \xrightarrow{\tau} \sigma^*E \xrightarrow{\pi^*E_1 = E} f^*E_2 = E_1 \xrightarrow{\sigma} E_2 \\
X & \xrightarrow{\sigma} X \xrightarrow{\pi} Y = X/\sigma \xrightarrow{\text{can. bun.}} S^4
\end{align*}$

The involution $\sigma : X \to X$ can be lifted to $E_1 \to E_1$ by choosing an isomorphism $\tau : E \to \sigma^*E$. Since $f$ has degree $k$, $\langle c_2(E_1), Y \rangle = \langle f^*c_2(E_2), Y \rangle = \langle c_2(E_2), f^*Y \rangle = \langle c_2(E_2), kS^4 \rangle = k$, and since $\pi$ is a branched double cover with codimension 2 branch set $\langle c_2(E), X \rangle = \langle \pi^*c_2E_1, X \rangle = \langle c_2E_1, \pi_*X \rangle = \langle c_2E_1, 2Y \rangle = 2k$.

The involution $\sigma$ on the $K3$ surface $X$ is an orientation preserving isometry $\tilde{\sigma}$ acts on the set $C$ of connections for $E \to X$ and acts on the group $\mathcal{G}$ of gauge transformations, via, $\tilde{\sigma}(\nabla) = \tilde{\sigma}\nabla\tilde{\sigma}^{-1}$ and $\tilde{\sigma}(g) = \tilde{\sigma}g\tilde{\sigma}^{-1}$ for $\nabla \in C$ and $g \in \mathcal{G}$. Since $\tilde{\sigma}$ is an isometry $\tilde{\sigma}$ acts on the set of anti-self-dual connections.

Lemma 5.2. (1) $\tilde{\sigma}$ acts $B$, and $\mathcal{M}$.

(2) Let $\mathcal{M}_{k,X}$ be the moduli space of equivalence classes of anti-self-dual connections with $c_2 = k$ on $E \to X$. Let $Y = X/\sigma$ be the orbit space of $\sigma$. Then

$$\dim \mathcal{M}_{2k,X} = 2 \dim \mathcal{M}_{k,Y}.$$ 

Proof of (2).

$$\dim \mathcal{M}_{2k,X} = 8(2k) - 3(1 + b^+(X)) \text{ since } b^+(X) = 3$$
$$= 2(8k - 3(4)) \text{ since } b^+(Y) = 1$$
$$= 2[8k - 3(1 + b^+(Y))]$$
$$= 2 \dim \mathcal{M}_{k,Y}$$
Theorem 5.3. The moduli space $\mathcal{M}_{k,Y}$ on the orbit space $Y$ can be identified one of the components of the $\sigma$-invariant moduli space $\mathcal{M}^\sigma_{2k,X}$ on $X$.

Proof. Consider the fundamental elliptic complex

$$0 \to \Omega^0(adP)^\sigma \overset{\nabla}{\to} \Omega^1(adP)^\sigma \overset{d^\sigma}{\to} \Omega^2(adP)^\sigma \to 0$$

where $\nabla \in \mathcal{M}^{\tilde{\sigma}}_{2k,X}$. By Lefschetz fixed point theorem

$$\dim \mathcal{M}^{\tilde{\sigma}}_{2k,X} = \text{ind}(\delta^\nabla + d^\nabla) = \frac{1}{2} \sum_{\sigma \in G} L(\sigma, D)$$

where $D : T(V_+ \otimes V_\sigma \otimes adP_c)^{\tilde{\sigma}} \to T(V_+ \otimes V_+)^{\tilde{\sigma}}$ is the twisted Dirac operator and $adP_c = adP \otimes_R C$. Let $\nabla : T(V_+ \otimes V_-) \to T(V_+ \otimes V_+)^{\tilde{\sigma}}$.

$$L(1, D) = P_1(adP \otimes C)[X] + 3\text{ind} \nabla$$

$$= 2k8 - \frac{3}{2} \left[2(\chi(Y) - \sigma(Y)) - (dx - d\sigma)\right]$$

$$= 2\left(8k - \frac{3}{2}(\chi(Y) - \sigma(Y)) - \frac{3}{2}(dx - d\sigma)\right)$$

$$= 2 \dim \mathcal{M}^{\tilde{\sigma}}_{2k,X} - 3(dx - d\sigma)$$

Since $\sigma$ acts trivially on $adP|X_R$. Thus $\dim \mathcal{M}^{\tilde{\sigma}}_{2k,X} = \frac{1}{2}[L(1, D) + L(\tilde{\sigma}, D)] = \dim \mathcal{M}_{k,Y}$.

Remark. When we compute $\text{ind}_{\sigma}(D)$ if $X$ is non orientable, twisted coefficients are used.

Let $\pi : X \to Y = X/\sigma$ be the projection and let $\mathcal{A}(E_1)$ be the space of connections with $c_2(E) = 2k$, and let $\mathcal{A}(E_1)$ be the space of the connection with $c_2(E_1) = k$ and $E = \pi^*E_1$.

As before choose a $\sigma$-invariant metric $g$ on $X$, then $\pi$ induces a singular metric $g_1$ on $Y$ such that $g = \pi^*(g_1)$. We would like to pull back the connections; $\pi^* : \mathcal{A}(E_1) \to \mathcal{A}(E)$ by setting for $\nabla \in \mathcal{A}(E_1)$ and $\varphi \in \Omega^0(E)$, and $v \in TX$, $\pi^*(\nabla)_v \phi = \nabla_{\pi_*v} \phi(\pi)$ since $E = \pi^*E$, $\phi(\pi)$ is a section of $E_1 \to Y$.

For any function $f \in C^\infty(X)$, we extend the definition of $\pi^*(\nabla)$ to be a connection on $E \to X$:

$$\pi^*(\nabla)_v(f\phi) = v(f)_\phi + f\pi^*\nabla_v \phi$$
where $\pi^*(\nabla_u)\phi = \hat{\nabla}_{\pi^* u}(\phi \pi)$. By construction $\pi^* \sigma^* = \pi^*$ and $\pi_* \sigma_* = \pi_*$, $\sigma(\pi^*(\nabla))_u \phi = \sigma(\pi^*(\nabla)_{\sigma u} \sigma \phi^{-1}) = \sigma(\nabla_{\pi^* u} (\pi) = \pi^*(\nabla)_u \phi$ is contained in the invariant subspace $A(E)^{\sigma}$. Thus we complete the proof of Theorem 5.3.

6. Donaldson Invariant on $K3$-surface

For any $\alpha \in H_2(X : Z)$, we choose an embedded oriented surface $\Sigma$ representing $\alpha$. For $\pi^* (\alpha) = \beta \in H_2(Y : Z)$, we choose an embedded oriented surface $\Sigma_1$ representing $\beta$. Let $N$ and $N_1$ be small tubular neighborhoods of $\Sigma$ and $\Sigma_1$ respectively. We may assume that $\pi(\Sigma) = \Sigma_1$. Let $r_\Sigma : B(X) \to B(N)$ be the restriction map and $r_{\Sigma_1} : B(Y) \to B(N_1)$.

For $A \in \mathcal{M}_{k,Y}$, consider the twisted Dirac operators over $N$ and $N_1$:

$$
\begin{array}{c}
\Gamma(V_{\Sigma}^- \otimes E) \\
\uparrow \pi^* \\
\Gamma(V_{\Sigma_1}^- \otimes E_1)
\end{array} \begin{array}{c}
\xrightarrow{\partial^*} \\
\xrightarrow{\partial A}
\end{array} \begin{array}{c}
\Gamma(V_{\Sigma}^+ \otimes E) \\
\uparrow \pi^* \\
\Gamma(V_{\Sigma_1}^+ \otimes E_1)
\end{array}
$$

For any $(\sigma \otimes s) \in \Gamma(V_{\Sigma_1}^- \otimes E_1)$.

$$
\partial_{\pi^* \lambda} \pi^*(\sigma \otimes s) = \partial_{\lambda}(\pi^* \sigma \otimes \pi^* s) = \partial_{\Sigma} \pi^* \sigma \otimes \pi^* s + \pi^* \sigma \otimes \pi^*(A)(\pi^* s) \\
= \pi^*(\partial_{\Sigma_1} \sigma \otimes s + \sigma \otimes A(S)) \\
= \pi^* \partial(A)(\sigma \otimes s), \text{ since } \pi_{\Sigma_1}^* \sigma = \partial_{\Sigma}(\pi^* \sigma).
$$

Thus the inclusion map $\pi^* : \mathcal{M}_{k,Y} \to \mathcal{M}_{2k,X}$ from $\mathcal{M}_{k,Y}$ into the invariant anti-self-dual connections of $\mathcal{M}_{2k,X}$ induces a bundle map on the determinant line bundles

$$
\begin{array}{c}
L_{\Sigma_1} \\
\downarrow \\
\mathcal{M}_{k,Y}
\end{array} \begin{array}{c}
\longrightarrow \\
\xrightarrow{\pi^*}
\end{array} \begin{array}{c}
L_{\Sigma} \\
\downarrow \\
\mathcal{M}_{2k,Y}
\end{array}
$$

There is a universal bundle $E$ over $X \times C^*/G$ with its Chern class $c_2(E) \in H^4(X \times C^*/G)$. For any class $\Sigma$ in $H_2(X)$, we have a map $\mu : H_2(X) \to H^2(C^*/G)$ which is defined by the slant product $\mu(\Sigma) = c_2(E)/\Sigma$.

**Lemma 6.1.** Given our two fold branched cover $\pi : X \to Y$, we have the following commutative diagram
Group Actions on Donaldson Invariants

\[ H_2(X) \xrightarrow{\mu_X} H^2(\mathcal{M}_{2k,X}) \]
\[ \downarrow \pi^* \quad \downarrow \pi^* \]
\[ H_2(Y) \xrightarrow{\mu_Y} H^2(\mathcal{M}_{k,Y}) \]

**Proof.** For any \( \alpha \in H_2(X) \), let \( \Sigma \) be an embedded oriented surface representing \( \alpha \). Since the inclusion map \( \pi^*: \mathcal{M}_{k,Y} \rightarrow \mathcal{M}_{2k,X} \) induces a bundle map and \( \mu_X(\alpha) = c_1(L_\Sigma) \) and \( \pi^*(\Sigma) = \Sigma_1 \), we have

\[
\pi^* \mu_X(\alpha) = \pi^*(c_1(L_\varepsilon)) = c_1(\pi^*(L_\Sigma)) = c_1(L_{\Sigma_1}) = c_1(L_{\pi^*\Sigma_1}) = \mu_Y(\pi^*(\alpha)).
\]

Given homology classes \( \alpha_1 \cdots \alpha_d \in H_2(X; \mathbb{Z}) \), we present them by embedded surfaces \( \Sigma_1 \cdots \Sigma_d \) in \( X \) in general position such that any triple intersections \( N_i \cap N_j \cap N_k \) of small tubular neighborhoods of \( \Sigma_i, \Sigma_j \), and \( \Sigma_k \) respectively are empty. There is a determinant line bundle \( L_\Sigma \) over \( B_X \) with a section whose zero set \( V_\Sigma \) is a codimension 2 submanifold of \( B_X \) and meets all of the moduli spaces \( \mathcal{M}_i \) for \( i \leq k \) transversally. If \( 4k > 3(1 + b_2^+ \end{array} \), then the intersection \( V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_k \) is compact.

**Definition 6.2.** For any homology classes \( \alpha_1, \cdots, \alpha_d \in H_2(X, \mathbb{Z}) \), the Donaldson invariant is defined to be

\[
q_{k,X}(\alpha_1, \cdots, \alpha_d) = < \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_d), [\mathcal{M}_{k,X}] >
\]

\[ = \sharp(V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_{k,X}) \]

where \( \sharp \) denotes a count with signs.

**Remark 6.3.** Suppose that \( g_1, g_2 \) are two different metrics on the base 4-manifold \( X \). The moduli space \( \mathcal{M}(g_1) \) and \( \mathcal{M}(g_2) \) of gauge equivalence classes of anti-self-dual connections with respects to the metrics \( g_1 \) and \( g_2 \) respectively differ by a boundary in \( B^* \). The pairing is independent of the choice of metric. Thus Donaldson invariants are smooth invariants.

Let \( \omega \) be the Kähler class in \( H^2(X; \mathbb{C}) \). By the Hodge index

\[
H^2_+ = \omega H^{2,0} \text{ and } b_2^+ = 1 + 2P_g.
\]

We fix the orientation of \( H^2_+ \) by \( -\omega \wedge \) (complex orientation of \( H^{2,0}(X; \mathbb{C}) \) which specifies the orientation of the moduli space. Let \( H \) be the hyperplane class in \( H_2(X; \mathbb{Z}) \) which is the Poincaré dual to \( \omega \). As a real manifold \( H \) is a compact Riemann surface. Over a small tubular
manifold $H$ in $S$, we have, for each connection $A$, a coupled Dirac operator \( \partial_H : \Gamma(V^- \otimes E) \rightarrow \Gamma(V^+ \otimes E) \). We may construct the determinant line bundle $L_H$ by using the index of the Dirac operators $\partial_H$ over $B_\Sigma$.

\[
L_H = \Lambda^{\max}(\ker \partial_H)^* \otimes \Lambda^{\max}(\ker \partial_H^*).
\]

The determinant line bundle $L_H$ descends to a bundle over $\mathcal{M}_H$. In fact $\mathcal{M}_H$ is a complex manifold and each connection $A$ defines the associated $\bar{\partial}$-operator $\bar{\partial}_A : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$ and defines a holomorphic structure on $E$. Thus we have the holomorphic line bundle $L_H \rightarrow \mathcal{M}_H$.

**Theorem 6.4** (Donaldson) (1) The determinant line bundle $L_H \rightarrow \mathcal{M}_H$ is an ample line bundle.

(2) $\mu(\alpha) = c_1(L_\Sigma) \in H^2(\mathcal{M}_X)$, where $L_\Sigma$ is the pull back of $L_H$ on $\mathcal{M}_X$ and $\Sigma$ is a representative of $[\alpha] \in H_2(X : \mathbb{Z})$.

In the $SU(2)$ universal bundle $E \rightarrow X \times \mathbb{C}/\mathcal{G}$, the orthogonal complement to orbits of $SU(2)$ gives the connection $A$. The curvature $F_A$ of $A$ is a horizontal 2-form with values in the Lie algebra $su(2)$. The tangent vectors are of type $(2,0)$, $(1,1)$ and $(0,2)$ in the tangent space $T_p(X) \times T_A(\mathcal{A}/\mathcal{G})$. Let $\rho \in H^2_{\text{deR}}(X : \mathbb{R})$ be the Poincaré dual to a homology class $\Sigma \in H_2(X : \mathbb{Z})$. The slant product $c_2(E)/\Sigma = \int_X c_2(E) \wedge \rho = \frac{1}{8\pi^2} \int_X \text{Tr}(F_A^2) \wedge \rho$ is the de-Rham representation of $\mu(\Sigma)$. If $a, b \in T_A B^* \cong \ker \delta A \subset \Omega^1(\mathcal{G}_E)$, then $\mu(\Sigma)(a \wedge b) = \frac{1}{8\pi^2} \int_X \text{Tr}(a \wedge b) \wedge \rho$. Let $X$ be a complex Kähler surface with Kähler form $\omega$. Let $H$ be the Poincaré dual to $\omega$. $\mu(H)(a \wedge b) = \frac{1}{8\pi^2} \int_X \text{Tr}(a \wedge b) \wedge \omega$ is the Kähler form on the moduli space $\mathcal{M}$ for the usual $L^2$-metric on $\mathcal{M}$. For the generic metric on $X$, the tangent bundle of $\mathcal{M}$ is the index bundle for $D_A : \Gamma(V^- \otimes V^+ \otimes \text{ad}P) \rightarrow \Gamma(V^+ \otimes V^+ \otimes \text{ad}P)$. If $A_1$ is a connection on $L_\Sigma \rightarrow \mathcal{M}_k$, then $c_1(L_\Sigma) = \frac{i}{2\pi} F_{A_1}$ and $F_{A_1}(a, b) = -i \int_X \text{tr}(a \wedge b) \wedge (PD \Sigma)$ where $a, b \in \Omega^1(\text{ad}P)$.

**Theorem 6.5.** Suppose that $L_\Sigma \rightarrow \mathcal{M}_k$ is the complex line bundle induced by a homology class $\Sigma \in H_2(X : \mathbb{Z})$ and $A$ is a connection of this bundle. Then the curvature is given by $F_A(a, b) = \frac{1}{4\pi i} \int_X \text{tr}(a \wedge b) \wedge \theta$, where $a, b \in \Omega^1(\text{ad}P)$ and $\theta = \text{Poincaré dual of } \Sigma$.

### 7. Involution on Donaldson Invariant

Recall that a $K3$-surface is a compact, simply connected complex surface with trivial canonical bundle. All $K3$-surfaces are diffeomorphic and Kählerian, but not necessarily biholomorphically equivalent. Some
$K3$-surface are elliptic surfaces, that is they admit a holomorphic map $\pi : X \to \mathbb{CP}^1$ whose generic fiber is an elliptic curve. $b_1^+ (X) = +3$, the stable range $4k > 3(1 + b^+)$, i.e. $k \geq 4$ the invariant $q_{k,s}$ is a multilinear function of degree $d = \frac{1}{2} \dim \mathcal{M}_{k,X} = 4k - 6$.

Let $Q$ be the quadratic form of the intersection form on $X$, and let $K$ be the linear function $K : H_2(X) \to \mathbb{Z}$ defined by the pairing $K(\alpha) = < c_1(X), \alpha >$ for any $\alpha$ in $H_2(X)$. The Donaldson invariants can be expressed as polynomials in $Q$ and $K : H_2(D_2(p,q)) \to \mathbb{Z}$ and have the form

$$(7.1) \quad q_{k,D_2(p,q)} = p, qQ^{[l]} + \sum_i a_i Q^{l-[i]} K^{2i} \quad \text{where} \quad l = \frac{d}{2} = \frac{1}{4} \dim \mathcal{M}_{k,X},$$

and $Q^{[l]} = \frac{1}{l!} Q^l$ and $Q^l (\sum_1 \cdots \sum_{2l}) = \frac{1}{(2l)!} \sum_\sigma Q(\sum_{\sigma_1} \sum_{\sigma_2}) \cdots Q(\sum_{\sigma_{2l-1}}, \sum_{\sigma_{2l}})$. 

We discussed the algebraic surface in section 2. Let $D_1(2,2q + 1)$ be obtained from $D_1$ by logarithmic transformation of multiplicities 2 and $2q + 1$. We introduce the results of Friedman and morgan.

**Theorem 7.2.** (Friedman, Morgan)

1. No two of the manifolds $D_1(2,2q + 1)(q = 0, 1 \cdots)$ are diffeomorphic, but all homeomorphic.

2. Let $D_2(p,q)$ be obtained from $K3$-surface by logarithmic transformations of multiplicities $p$ and $q$. Then the product $pq$ is a smooth invariant. In particular, no two of $D_2(1,2k + 1)$ are diffeomorphic.

**Example 7.3.** Let $X$ be a $K3$-surface. Let $E \to X$ be an $SU(2)$ vector bundle with $c_2(E) = 4$. The moduli space $\mathcal{M}_{4,x}$ of anti-self-dual connections has dimension 20. Let $Q$ be the quadratic form of the intersection on $X$.

By (7.1) we have, for $z_1, \cdots, z_{10} \in H_2(X : \mathbb{Z})$ such that $z = z_i (i = 1, \cdots, 10)$, $Q(z, z) = 2$

$q_{4,x}(z_1, \cdots z_{10}) = Q^{(5)}(z_1, \cdots, z_{10})$

$= \frac{1}{5!2^5} \sum_{\sigma \in S_{10}} Q(z_{\sigma_1} z_{\sigma_2}) \cdots Q(z_{\sigma_9} z_{\sigma_{10}})$

$= 1$.

This result also proved by Fintushel and Stern for homology $K3$-surface.

Suppose that $\alpha_1 \cdots \alpha_d \in H^2(X : \mathbb{Z})$. We use the Lemma 6.1 to compute Donaldson invariant.

$< \mu_X(\alpha_1) \cdots \mu_X(\alpha_d), \pi^* \mathcal{M}_{k,Y} > = < \pi^*(\mu_X(\alpha_1) \cdots \pi^* \mu_X(\alpha_d)), \mathcal{M}_{k,Y} >$
Thus we have a theorem.

**Theorem 7.4.** Let \( \sigma \) be an anti-holomorphic involution on a K3-surface \( X \). If \( Y = X/\sigma \) is the orbit space, then the Donaldson invariant \( q_{k,Y} \) can be computed by the pairing the invariant moduli space \( M_{2k,X}^{\sigma} \) and the cohomology classes in \( M_{2k,X} \).

For a K3-surface \( X \), let \( O_Q \) be the isometry group of the intersection form \( Q \) on the integral homology \( H_2(X) \) and the homomorphism

\[
h : \text{Diff}(X) \to O_Q.
\]

The isometry group \( O_Q \) contains an index 2 subgroup \( O_Q^+ \) consisting of transformations which preserve the orientations of the positive part \( H_2^+(X) \cong \mathbb{Z}^3 \) of \( H_2 \). Since \(-1\) does not lie in \( O_Q^+ \), there is a splitting \( O_Q = O_Q^+ \oplus O_Q^- \).

**Theorem 7.5.** (Donaldson and Matumoto) The image of \( h : \text{Diff}(X) \to O_Q \) is the subgroup \( O_Q^+ \).

**Theorem 7.6.** Let \( \sigma \) be an anti-holomorphic involution on a K3 surface \( X \), then

\[
\sigma^* q_{2k,X} = q_{2k,X} : S^{2d}(H_2(X, \mathbb{Z})) \to \mathbb{Z}
\]

**Proof.** If \( \sigma \) is an anti-holomorphic involution on a K3-surface \( X \), then \( \sigma \) is an orientation preserving diffeomorphism. By Donaldson and Matumoto Theorem, \( \sigma^* \) is an isometry on \( H^2(X : \mathbb{Z}) \) and preserves the orientation of the positive part of the second cohomology group \( H^2(X : \mathbb{Z})^+ = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \). Thus \( \sigma^* \) preserves the Donaldson invariants on \( S^{2d}(H_2(X : \mathbb{Z})) \).

**References**


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