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DIRECT SUM AND SIMULTANEOUS EQUATIONS IN THE PREDUAL OF A DUAL ALGEBRA

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Dedicated to Professor Younki Chae on the occasion of his sixtieth birthday.

In this paper we study the relationship between direct sum and the systems of simultaneous equations in the predual of a dual algebra. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. It is well-known that the weak*-topology coincides with the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ (cf. [4]). A unital subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ that is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ is called a *dual algebra.* In the past fourteen years several operator theorists have been studying the problem of solving systems of simultaneous equations in the predual of dual operator algebras and the results gained thereby have applied to invariant subspaces, dilation theory, and reflexivity of single operator in $\mathcal{L}(\mathcal{H})$ (cf. [2]).

Furthermore, the theory of dual algebras is closely related to properties $(\mathbf{A}_{m,n})$ which will be introduced in the following definition, where m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. In particular, Apostol-Bercovici-Foias-Pearcy [1] studied the relationship between dual algebras and properties $(\mathbf{A}_{m,n})$, which is the main tool of this work.

The notation and terminology employed herein agree with those in [2]. The class $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace-class operators on $\mathcal{H}$

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equipped with the trace norm. The dual algebra $\mathcal{A}$ can be identified with the dual space of

$$\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}, \tag{1}$$

where $^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing

$$< T, [L]_{\mathcal{A}} >= tr(TL), T \in \mathcal{A}, [L] \in \mathcal{Q}_{\mathcal{A}}. \tag{2}$$

For a brief notation, we write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathcal{H}$, we denote a rank one operator $(x \otimes y)(u) = (u, y)x$ for all u in $\mathcal{H}$.

Definition 1. Suppose m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], 0 \leq i < m, 0 \leq j < n, \tag{3}$$

where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m'} \{y_j\}_{0 \leq j < n}$ consisting of a pair of sequence of vectors from $\mathcal{H}$. For a brief notation, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$.

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{A}_T$ the dual algebra generated by T. N is the set of all natural numbers. For $k \in \mathbf{N}$ and $T \in \mathcal{L}(\mathcal{H})$, we denote

$$\mathcal{H}^{(k)} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{(k)} \text{ an } T^{(k)} = \underbrace{T \oplus \cdots \oplus T}_{(k)}. \tag{4}$$

Let $\mathcal{H}_i$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$ be a dual algebra, $i = 1, 2, \cdots, n$. Suppose $\widetilde{\mathcal{H}} = \oplus_{i=1}^{n} \mathcal{H}_i$. Then we denote the direct sum of dual algebras $\mathcal{A}_i, i = 1, 2, \cdots, n$, and denote

$$\widetilde{\mathcal{A}} = \oplus_{i=1}^{n} \mathcal{A}_i = \{\oplus_{i=1}^{n} T_i \in L(\widetilde{\mathcal{H}}) | T_i \in \mathcal{A}_i, i = 1, 2, \cdots, n\}. \tag{5}$$

And we have the following lemma.

Lemma 2. *Let $\mathcal{H}_i$ be a separable, infinite dimensional, complex Hilbert space. Suppose that $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$ is a dual algebra, $i = 1, 2, \cdots, n$, with its predual $\mathcal{Q}_{\mathcal{A}_i}$. Then $\widetilde{\mathcal{A}} \subset \mathcal{L}(\widetilde{\mathcal{H}})$ is a dual algebra with its predual $\oplus_{i=1}^{n} \mathcal{Q}_{\mathcal{A}_i}$ under the duality*

$$< \oplus_{i=1}^{n} T_i, \oplus_{i=1}^{n} [L_i]_{\mathcal{A}_i} >= \sum_{i=1}^{n} < T_i, [L_i] > \tag{6}$$

and the norm on $\oplus_{i=1}^n \mathcal{Q}_{\mathcal{A}_i}$ is the norm that accures to it as a linear manifold in $\widetilde{\mathcal{A}}^*$. In particular, $[(\oplus_{i=1}^n x_i) \otimes (\oplus_{i=1}^n y_i)]$ can be identified with $\oplus_{i=1}^n [x_i \otimes y_i]_{\mathcal{A}_i}$.

Proof. First we shall claim that $\widetilde{\mathcal{A}}$ can be considered as a subspace of $\mathcal{L}(\widetilde{\mathcal{H}})$ under the weak*-topology on $\mathcal{L}(\mathcal{H})$. Namely, we shall claim the closedness of $\widetilde{\mathcal{A}}$. To do so, let $\oplus_{i=1}^n T_i^{(\alpha)}$ be a net converging to an operator $R \in \mathcal{L}(\widetilde{\mathcal{H}})$ under the weak*-topology on $\mathcal{L}(\widetilde{\mathcal{H}})$. Then

$$(7) \qquad \sum_{k=1}^{\infty} (\oplus_{i=1}^n T_i^{(\alpha)} \tilde{x}^{(k)}, \tilde{u}^{(k)} \longrightarrow \sum_{k=1}^{\infty} (R\tilde{x}^{(k)}, \tilde{u}^{(k)})$$

for any square summable sequences $\{\tilde{x}^{(k)}\}_{k=1}^{\infty}$ and $\{\tilde{u}^{(k)}\}_{k=1}^{\infty}$ in $\widetilde{\mathcal{H}}$. Let us write

$$(8) \qquad R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}$$

relative to $\widetilde{\mathcal{H}}$. Let us denote

$$(9a) \qquad \tilde{x}^{(k)} = x_1^{(k)} \oplus \cdots \oplus x_n^{(k)}$$

and

$$(9b) \qquad \tilde{u}^{(k)} = u_1^{(k)} \oplus \cdots \oplus u_n^{(k)}.$$

Now we take square summable sequences $\{x_i^{(k)}\}_{k=1}^{\infty}$ in $\mathcal{H}_i$ and $\{y_j^{(k)}\}_{k=1}^{\infty}$ in $\mathcal{H}_j$, $1 \le i, j \le n$. Let us set

$$(10a) \qquad \tilde{x}_i^{(k)} = \overbrace{\underbrace{0 \oplus \cdots \oplus 0}_{(i-1)} \oplus x_i^{(k)} \oplus 0 \oplus \cdots \oplus 0}^{(n)},$$

and

$$(10b) \qquad \tilde{y}_j^{(k)} = \overbrace{\underbrace{0 \oplus \cdots \oplus 0}_{(j-1)} \oplus y_j^{(k)} \oplus 0 \oplus \cdots \oplus 0}^{(n)},$$

Substitute $\{\tilde{x}_i^{(k)}\}_{i=1}^{\infty}$ and $\{\tilde{y}_j^{(k)}\}_{k=1}^{\infty}$ to (7), and we have

$$\begin{aligned}(11)\qquad &\sum_{k=1}^{\infty}((\oplus_{\ell=1}^{n} T_\ell^{(\alpha)})\tilde{x}_i^{(k)}, \tilde{y}_j^{(k)}) \\ &\to \sum_{k=1}^{\infty}(R_{1i}x_i^{(k)} \oplus R_{2i}x_i^{(k)} \oplus \cdots \oplus R_{ni}x_i^{(k)}, \tilde{y}_j^{(k)}),\end{aligned}$$

for any $i, j = 1, 2, \cdots, n$. It is easy to show that

$$(12)\qquad R_{ji} = 0, i \neq j.$$

Hence $R = \oplus_{i=1}^{n} R_{ii}$. Furthermore, we have that

$$\begin{aligned}(13)\qquad &\sum_{k=1}^{\infty}(T_i^{(\alpha)}x_i^{(k)}, y_i^{(k)}) \\ &\to \sum_{k=1}^{\infty}(R_{ii}x_i^{(k)}, y_i^{(k)}), 1 \leq i \leq n.\end{aligned}$$

Since $\mathcal{A}_i$ is weak*-closed, $R_{ii} \in \mathcal{A}_i$. So $R \in \oplus_{i=1}^{n}\mathcal{A}_i$. Therefore $\widetilde{\mathcal{A}}$ is a dual algebra in $\mathcal{L}(\widetilde{\mathcal{H}})$.

Now consider the direct sum

$$(14)\qquad \oplus_{i=1}^{n} \mathcal{Q}_{\mathcal{A}_i} = \{\oplus_{i=1}^{n}[L_i]_{\mathcal{A}_i} \mid [L_i]_{\mathcal{A}_i} \in \mathcal{Q}_{\mathcal{A}_i}\}$$

of Banach spaces $\mathcal{Q}_{\mathcal{A}_i}$, $1 \leq i \leq n$, with the usual direct sum norm.

The following idea comes from the proof of [1, Lemma 1.2]. For $\oplus_{i=1}^{n}T_i \in \widetilde{\mathcal{A}}$ and $\oplus_{i=1}^{n}[L_i]_{\mathcal{A}_i} \in \oplus_{i=1}^{n}\mathcal{Q}_{\widetilde{\mathcal{A}}}$, we define

$$(15)\qquad < \oplus_{i=1}^{n}T_i, \oplus_{i=1}^{n}[L_i]_{\mathcal{A}_i} >= \sum_{i=1}^{n} < T_i, [L_i]_{\mathcal{A}_i} > .$$

Then it is easy to show that $< \cdot, \oplus_{i=1}^{n}[L_i]_{\mathcal{A}_i} >$ defines a bounded linear functional on $\widetilde{\mathcal{A}}$, which we may define by $\oplus_{i=1}^{n}[L_i]$. We define $\| \oplus_{i=1}^{n} [L_i]\|$ to be the norm of this linear functional. Since $\oplus_{i=1}^{n}[L_i]$ is ultraweakly continuous on $\oplus_{i=1}^{n}\mathcal{A}_i$, by [3, Problem 15.J] $\oplus_{i=1}^{n}[L_i]$ corresponds to an element of the predual $\mathcal{Q}_{\widetilde{\mathcal{A}}}$.

On the other hand, if $[L] \in \mathcal{Q}_{\widetilde{\mathcal{A}}}$, we write

$$(16)\qquad L = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \in \mathcal{L}(\widetilde{\mathcal{H}}).$$

Furthermore, since $\widetilde{\mathcal{A}} \subset \mathcal{L}(\widetilde{\mathcal{H}})$, we may define a linear functional on $\widetilde{\mathcal{A}}$ such that

$$< \oplus_{i=1}^{n} A_i, [L] >= tr(A_{i_0}, L_{i_0 i_0}). \tag{17}$$

Letting i_0 range over the set $\{1, \cdots, n\}$, we obtain an element $\oplus_{i=1}^{n}[L_i]$ corresponding to $[L]$ and

$$< \oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n}[L_i] >= \sum_{i=1}^{n} < A_i, [L_i] > . \tag{18}$$

Finally, for any $\oplus_{i=1}^{n} T_i \in \widetilde{\mathcal{A}}$, we have

$$\begin{aligned} (19)\quad < \oplus_{i=1}^{n} T_i, [(\oplus_{i=1}^{n} x_i) \otimes (\oplus_{i=1}^{n} y_i)] > &= (\oplus_{i=1}^{n} T_i x_i, \oplus_{i=1}^{n} y_i) \\ &= \sum_{i=1}^{n} (T_i x_i, y_i) \\ &= \sum_{i=1}^{n} < T_i, [x_i \otimes y_i]_{\mathcal{A}_i} > \\ &= < \oplus_{i=1}^{n} T_i, \oplus_{i=1}^{n} [x_i \otimes y_i]_{\mathcal{A}_i} > . \end{aligned}$$

Hence $[(\oplus_{i=1}^{n} x_i) \otimes (\oplus_{i=1}^{n} y_i)]$ can be identified with $\oplus_{i=1}^{n}[x_i \otimes y_i]$. The proof is complete.

The following lemma is a generalization of [2, Proposition 2.04].

Lemma 3. *If $\mathcal{A}$ is any algebra with property $(\mathbf{A}_{m,n})$ for some $1 \le m, n \le \aleph_0$ and $\mathcal{B}$ is any subalgebra of $\mathcal{A}$, then $\mathcal{B}$ has the same property.*

Proof. The proof is similar to [2, Proposition 2.04].

The following theorem should be compared with [1, Proposition 1.3] and [2, Proposition 2.055].

Theorem 4. *Suppose $m, n \in \mathbf{N}$. Let $\mathcal{A}_i$ be a dual algebra, $1 \le i \le n$. Then $\mathcal{A}_i$ has property $(\mathbf{A}_{m,n})$ for any $i = 1, \cdots, n$, if and only if $\oplus_{i=1}^{n} \mathcal{A}_i$ has property $(\mathbf{A}_{m,n})$.*

Proof. Let $\oplus_{k=1}^{n}[L_{ij}^{(k)}] \in \oplus_{k=1}^{n} Q_{\mathcal{A}_k}$. Then there exist $x_i^{(k)}, y_j^{(k)} \in \mathcal{H}_k$, $1 \le i \le m$, $1 \le j \le n$, such that

$$[L_{ij}^{(k)}]_{\mathcal{A}_k} = [x_i^{(k)} \otimes y_j^{(k)}]_{\mathcal{A}_k}. \tag{20}$$

Now let us set

$$\tilde{x}_i = x_i^{(1)} \oplus x_i^{(2)} \oplus \cdots \oplus x_i^{(n)} \tag{21a}$$

and

$$(21b) \qquad y_j = x_j^{(1)} \oplus x_j^{(2)} \oplus \cdots \oplus x_j^{(n)}.$$

Then $\tilde{x}_i \in \widetilde{\mathcal{H}}$ and $\tilde{y}_i \in \widetilde{\mathcal{H}}$, $1 \leq i, j \leq n$. Furthermore according to Lemma 2 we have

$$(22) \qquad \oplus_{k=1}^{n} [L_{ij}^{(k)}]_{\mathcal{A}_i} = \oplus_{k=1}^{n} [x_i^{(k)} \otimes y_j^{(k)}] = [\tilde{x}_i \otimes \tilde{y}_j].$$

The converse implication follows from Lemma 3 and the proof is complete.

References

[1] C. Apostol, H. Bercovici, C. Foias and C. Pearcy, *Invariant subspaces, dilation theory, and structure of the predual of a dual algebra I*, J. Funct. Anal., 63(1985), 369-404.

[2] H. Bercovici, C., Foias and C. Pearcy, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series, no. 56, A.M.S., Providence, R.I., 1985.

[3] A. Brown and C. Pearcy, *Introduction to operator theory I*, Elements of functional analysis, Springer-Verlag, New York, 1977.

[4] J. Dixmier, *Von Neumann algebras*, North-Holland Publishing Company, Amst. New York, Oxford 1969.

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