ON THE UNICELLULARITY OF SOME OPERATORS

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Dedicated to Professor Younki Chae on his 60th birthday

I. Introduction

The investigation of invariant subspaces is the first step in the attempt to understand the structure of bounded linear operators. We will investigate bounded linear operators on Hilbert spaces which have the simplest possible invariant subspace structure.

It is well known that the Donoghue weighted shift operator and the Volterra operator are unicellular, and there are several approaches to prove it. In an effort to extend this result we would like to determine when strictly lower triangular operators on $l^2$ are unicellular. And we are going to apply the result to study the unicellularity of a discrete Volterra operator under certain condition.

We introduce some definitions. Let $H$ be a Hilbert space and $A$ an operator on $H$. Let $M$ denote a subspace of $H$, that is, a closed linear manifold of $H$. The subspace $M$ is invariant under the operator $A$ if $Ax \in M$ for every $x \in M$. The collection of all subspaces of $H$ invariant under $A$ is denoted by $Lat A$. The operator $A$ is unicellular if the collection $Lat A$ is totally ordered by inclusion. If $K$ is a subset of $H$, the span of $K$ is the smallest subspace containing $K$ and denoted by $span K$. If $x \in H$, then $span\{x, Ax, A^2x, \ldots\}$ is easily seen to be invariant under $A$. The vector $x$ is cyclic for $A$ if $span\{x, Ax, A^2x, \ldots\} = H$ and $M$ is a cyclic subspace for $A$ if $span\{x, Ax, A^2x, \ldots\} = M$.

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II. Unicellularity of triangular operators

Let $A$ be a bounded operator with $\|A\| < 1$ on $l^2$ and let $\{e_0, e_1, \ldots\}$ denote the standard basis for $l^2$. Let $x$ be a column vector in $l^2$. Then $A^n x$ is a column vector in $l^2$ for $n = 0, 1, 2, \ldots$. Then we have an infinite matrix $[x, Ax, A^2x, \cdots]^t$ which will be denoted by $S_x(A)$. The matrix $S_x(A)$ is a bounded linear transformation on $l^2$.

Let $A$ be a bounded operator with $\|A\| < 1$ on $l^2$ represented by a strictly lower triangular matrix. Let $M_n$ be the subspace span$\{e_n, e_{n+1}, \ldots\}$ for every $n = 0, 1, 2, \ldots$. Then every $M_n$ is invariant under $A$ and $\{M_n | n = 0, 1, 2, \ldots\} \cup \{0\}$ is totally ordered by inclusion; $l^2 = M_0 \supset M_1 \supset M_2 \supset \cdots$. $A$ is unicellular if its only invariant subspaces are $\{0\}$ and $M_n$, $n = 0, 1, 2, \ldots$, i.e. the collection Lat $A$ of all subspaces of $l^2$ invariant under $A$ is $\{\{0\}\} \cup \{M_n\}_{n=0}^{\infty}$. Let $M$ be a subspace of $l^2$ and $M^* = \{\sum_{n=0}^{\infty} c_n e_n : \sum_{n=0}^{\infty} c_n e_n \in M\}$. If we let $x_N = (0, \cdots, 0, 1, x_{N+1}, \cdots) \in l^2$ and $M_{\xi N} = \text{span}\{x_N, Ax_N, A^2x_N, \cdots\}$ then $M_{\xi N}^\perp = (\text{Ker}(S_{\xi N}(A)))^*$ and always $M_{\xi N}^\perp \subset (\text{Ker}S_{\xi N}(A))^*$.

Lemma 2.1. Let $A$ be a strictly lower triangular operator on $l^2$ and let $x \in l^2$. Then $(U^N A U^N)^n U^N x = U^N A^N P_N x$ for every $n, N = 0, 1, 2, \ldots$, where $U$ is the unilateral shift on $l^2$ and $P_N$ the orthogonal projection on $M_N$.

Proof. Let $N$ be a nonnegative integer. For $n = 0$, $U^N x = U^N P_N x$. We assume that $(U^N A U^N)^n U^N x = U^N A^n P_N x$. Then

$$(U^N A U^N)^{n+1} U^N x = (U^N A U^N)(U^N A U^N)^n U^N x = (U^N A U^N) U^N A^n P_N x,$$
by induction hypothesis$
= U^N A P_N A^n P_N x$
$= U^N A A P_N x$, since $A$ is strictly lower triangular$
= U^N A^{n+1} P_N x$.

Lemma 2.2. Let $A$ be a strictly lower triangular operator with $\|A\| < 1$ on $l^2$ and let $x_N = (0, \cdots, 0, 1, x_{N+1}, \cdots) \in l^2$ for every $N = 0, 1, 2, \cdots$. $M_N$ is a cyclic subspace for $A$, i.e. $M_N = M_{\xi N}$ if and only if $S_{U^N_{\xi N}}(U^N A U^N)$ is one-to-one.

Proof. $M_N$ is a cyclic subspace for $A$, i.e. $M_N = M_{\xi N}$, if and only if
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\((\text{Ker} S_{\xi_N}(A))^* = M_N^{1}\) if and only if \((\text{Ker} S_{\xi_N}(A))^* \subset M_N^{1}\). Let \(y \in l^2\).

\[ S_{\xi_N}(A)\bar{y} = \begin{bmatrix} \bar{x}_N \\ A\bar{x}_N \\ A^2\bar{x}_N \\ \vdots \end{bmatrix} = \begin{bmatrix} \bar{y}_0 \\ \bar{y}_1 \\ \bar{y}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} N \\ 0 \cdots 0 & 1 & x_{N+1} & \cdots \\ 0 \cdots 0 & 0 & * & \cdots \\ 0 \cdots 0 & 0 & 0 & * & \cdots \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{y}_0 \\ \bar{y}_1 \\ \bar{y}_2 \\ \vdots \end{bmatrix} \]

\(S_{\xi_N}(A)\bar{y} = 0\) if and only if

\[ 0 = \begin{bmatrix} 1 & x_{N+1} & \cdots \\ 0 & * & \cdots \\ 0 & 0 & * & \cdots \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{y}_N \\ \bar{y}_{N+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \text{U}^{*N}x_N \\ \text{U}^{*N}A\bar{x}_N \\ \text{U}^{*N}A^2\bar{x}_N \\ \vdots \end{bmatrix} \begin{bmatrix} \bar{y}_N \\ \bar{y}_{N+1} \\ \vdots \end{bmatrix} \]

By Lemma 2.1, \(\text{U}^{*N}A^n\bar{x}_N = \text{U}^{*N}A^n\text{P}_N\bar{x}_N = (\text{U}^{*N}A\text{U}^N)^n \text{U}^{*N} \bar{x}_N\) for each \(n = 0, 1, 2, \ldots\). Hence \((\text{Ker} S_{\xi_N}(A))^* \subset M_N^{1}\) if and only if

\[ \begin{bmatrix} \text{U}^{*N}x_N \\ \text{U}^{*N}A\bar{x}_N \\ \text{U}^{*N}A^2\bar{x}_N \\ \vdots \end{bmatrix} = \begin{bmatrix} \text{U}^{*N}x_N \\ (\text{U}^{*N}A\text{U}^N)^n \text{U}^{*N} \bar{x}_N \\ (\text{U}^{*N}A\text{U}^N)^2 \text{U}^{*N} \bar{x}_N \\ \vdots \end{bmatrix} = S_{\text{U}^{*N}x_N}(\text{U}^{*N}A\text{U}^N) \]

is one-to-one.

**Theorem 2.3.** Let \(A\) be a strictly lower triangular operator with \(\|A\| < 1\) on \(l^2\). Then \(A\) is unicellular if and only if for any \(x = (1, x_1, \cdots) \in l^2\), \(S_x(\text{U}^{*N}A\text{U}^N)\) is one-to-one for every \(N = 0, 1, 2, \ldots\).

**Proof.** If \(A\) is unicellular, then \(\text{Lat} A = \{\{0\}\} \cup \{M_n\}_{n=0}^\infty\). Let \(x = (1, x_1, \cdots) \in l^2\) and \(N\) be a fixed nonnegative integer. Then \(\text{U}^N x = (0, \cdots, 0, 1, x_1, \cdots) \in l^2\) and \(M_{\text{U}^N x} = \text{span}\{\text{U}^n x, \text{A}\text{U}^n x, \cdots\}\) is an invariant subspace of \(l^2\) for \(A\) and \(M_{\text{U}^N} = M_n\) for some \(n\). Clearly, \(M_{\text{U}^N x} = M_N\). Hence \(M_N\) is a cyclic space for \(A\) and \(\text{U}^{N} x = x\). Therefore \(S_x(\text{U}^{*N}A\text{U}^N)\) is one-to-one.

Conversely, we assume that for any \(x = (1, x_1, \cdots) \in l^2\), \(S_x(\text{U}^{*N}A\text{U}^N)\) is one-to-one for every \(N = 0, 1, 2, \ldots\). Let \(M\) be an invariant subspace of
We need to show that \( \mu \) is \( \{0\} \) or \( M_n \) for some nonnegative integer \( n \). Assume that \( M \neq \{0\} \). Let \( N \) be the least index of non-zero entries of all elements of \( M \). Then \( 0 \leq N < \infty \), \( M \subseteq M_N \) and \( M \) contains a vector \( x \) of form \((0, \cdots, 0, 1, x_{N+1}, \cdots)\). From the assumption \( S_{U^N}^* (U^N A U^N) \) is one-to-one. Hence \( M_N \) is a cyclic subspace for \( A \), i.e. \( M_N = M_x \subseteq M \).

Hence \( M_N = M \).

Now we need some properties of strictly upper triangular matrices in order to determine whether they are one-to-one.

**Theorem 2.4.** Let \( T \) and \( S \) be bounded operators on a Hilbert space \( H \) represented by upper triangular matrices with respect to a fixed orthonormal basis. Assume that all diagonal entries of \( T \) are non-vanishing, and that all diagonal entries of \( S \) are 0. If \( T \) is invertible and \( S \) is compact, then \( T + S \) is one-to-one.

**Proof.** Since \( T \) is invertible, \( \text{Ker} T = \{0\} \) and \( \text{Ker} T^* = \{0\} \). Then \( \text{index} (T) = \text{dim} \text{Ker} T - \text{dim} \text{Ker} T^* = 0 \). Since \( S \) is compact, \( \text{index} (T + S) = \text{index} (T) = 0 \) by the index theory in [3]. Since \( T + S \) is an upper triangular matrix with non-zero diagonals, \( \text{Ker} (T + S)^* = \{0\} \). Hence \( 0 = \text{index} (T + S) = \text{dim} \text{Ker} (T + S) - \text{dim} \text{Ker} (T + S)^* \). So \( \text{dim} \text{Ker} (T + S) = 0 \). Therefore \( T + S \) is one-to-one.

**Corollary 2.5.** Let \( C \) is a strictly upper triangular matrix on a Hilbert space \( H \) with respect to an orthonormal basis \( \{e_n\}_{n=0}^{\infty} \). Let

\[
C = \begin{bmatrix}
A & B \\
0 & D
\end{bmatrix}
\]

where \( H = H_n \oplus H_n^\perp \) and \( H_n = \text{span}\{e_0, \cdots, e_n\} \). If \( D \) is a compact operator, then \( I + C \) is one-to-one.

**III. Discrete Volterra Operators**

In [2], Chan showed the unicellularity of a discrete Volterra operator under certain condition. We now use the previous result to show the unicellularity of a discrete Volterra operator. In this case we have a little different condition from the result in [2].

**Definition 3.1.** [2] Suppose \( t_n > 0 \) for all \( n \), and \( \sum_n t_n^2 = 1 \). Then the
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operator $V : l^2 \rightarrow l^2$ defined by

$$
\begin{bmatrix}
0 & t_0 t_1 & t_0 t_2 & \cdots \\
0 & t_1 t_2 & \cdots \\
& 0 & \cdots \\
& & & \ddots
\end{bmatrix}
$$

is called a \textit{Discrete Volterra operator}.

The adjoint $V^*$ of the Volterra operator is defined by

$$
\begin{bmatrix}
0 & t_0 t_1 & 0 & 0 \\
t_0 t_2 & t_1 t_2 & 0 & 0 \\
& & & \ddots
\end{bmatrix}
$$

Remarks. The adjoint $V^*$ of the discrete Volterra operator $V$ satisfies the following:

i) $\pi_0(V^*) = \{0\}$. (Here $\pi_0(A)$ denotes the point spectrum of $A$, i.e. $\{z \in C : Ker(A - zI) = \{0\}\}$).

ii) $Ker(V^*) = \{0\}$, so $V^*$ is one-to-one.

iii) $\sum_{ij} |V_{ij}^*|^2 < 1$, so $V^*$ is a Hilbert-Schmidt operator.

We note that $V$ is similar to a Donoghue weighted shift operator if $t_{n+1}/t_n \leq r$ and $r < 1/2$, see [2]. Thus $V$ is unicellular. Chan gave a direct proof that a discrete Volterra operator $V$ is unicellular if $t_{n+1}/t_n \leq r < \sqrt{2} - 1$ in [2]. In this section, we use the method of section 2, to give a direct proof of the unicellularity of $V^*$ when $t_{n+1}/t_n \leq r$ and $0 < r \leq 0.43$. Note that $\sqrt{2} - 1 < 0.43$.

A computation establishes:

Lemma 3.2.

i) $V^*_{n,ij} = \begin{cases} 
\sum t_j t_{k_1} t_{k_2} \cdots t_{k_{n-1}} t_i = V^n_{ji} & \text{if } i \geq n \text{ and } i > j \\
0 & \text{otherwise}
\end{cases}$

where $j < k_1 < \cdots < k_{n-1} < i$

ii) The $(i,j)$ entry of the matrix $\frac{V^*}{i_0 t_1^{2} \cdots t_{n-1}^{2} t_n}$ is

$$
\left[\frac{V^*}{i_0 t_1^{2} \cdots t_{n-1}^{2} t_n}\right]_{ij} = \frac{\sum_{j<k_1<\cdots<k_{n-1}<t_j t_{k_1}^{2} \cdots t_{k_{n-1}}^{2} t_i}}{i_0 t_1^{2} \cdots t_{n-1}^{2} t_n} = \frac{V^n_{ji}}{V^n_{0n}}.
$$
Lemma 3.3. For $n \geq 1$ and $j \geq 1$, if $t_{n+1}/t_n \leq r < 1$, then

1. $\frac{V_{n}^{0}+t_j}{V_{n+1}^{0}+t_{n+1}} < r(1 + r^{2j} + r^{4j} + \ldots + r^{2(n-1)j}) < \frac{r}{1-r^2j}.$

2. $\frac{V_{n}^{j}}{V_{n+1}^{j}+t_{n+1}+t_j} < r(1 + r^{2j} + r^{4j} + \ldots + r^{2(n-1)j}) < \frac{r}{1-r^2j}.$ In particular, if $j = 2$ then we have more accurate estimation.

3. $\frac{V_{n}^{0}+t_j}{V_{n+1}^{0}+t_{n+1}} < r(1 + r^{4j} - r(n-1)1 - r^{2j}) + r^{2} - r^{4} + \ldots + r^{4(n-1)}, 1 - r^{2j}.$

4. $\frac{V_{n}^{j}}{V_{n+1}^{j}+t_{n+1}+t_j} \leq r^{2i}$ If $R_j = 1/(1 - r^{2j})$ for $j \geq 1$, then

Moreover, $\frac{V_{n}^{j}}{V_{n+1}^{j}+t_{n+1}+t_j} < r^{2i} + (rR_1)(rR_2) \ldots (rR_j) < r^{2i} + (rR_1)^{q-1}(rR_j)^{p-1}$

where $F_q$ is constant and depended only on $q$.

Proof. Let $n \geq 1, j \geq 1$ and $t_{n+1}/t_n \leq r < 1$ for all $n$.

\[
\frac{V_{n}^{0}+t_j}{V_{n+1}^{0}+t_{n+1}+t_j} = \sum_{0 < k_1 < \ldots < k_{n-1} < n + j} \frac{t_1t_2 \ldots t_{k-1}t_{k+1} \ldots t_{n-1}t_{n}}{b_0 + t_1t_2 \ldots t_{k-1}t_{k+1} \ldots t_{n-1}t_{n+1}}.
\]

where $0 < k_1 < \ldots < k_{n-m} < n + j - m$, and $m = 1, \ldots, n - 1$. Since

\[
\sum_{0 < k_1 < \ldots < k_{n-m} < n + j - m} t_{k_1}t_2 \ldots t_{k_{n-m}}t_{m+1}^{2} \leq \sum_{0 < k_1 < \ldots < k_{n-1} < n-1} t_{k_1}t_2 \ldots t_{k_{n-1}}t_{n-1}^{2}.
\]
for all $m = 1, \ldots, n - 1$,

$$\frac{V^{n_0 n + i}}{V^{n_0 n + i + 1}} < \frac{t_{i+1} \cdots t_{i+2} \cdots t_{n} - 2 \cdot t_{n+1} - 1 \cdot t_{n+2} \cdots t_{n}}{t_{i+1} \cdots t_{i+2} \cdots t_{n} - 2 \cdot t_{n+1} - 1 \cdot t_{n+2} \cdots t_{n} + t_{i} \cdots t_{n+1} - 1 \cdot t_{n+2} \cdots t_{n} + 1 \cdot t_{n+1} - 1 \cdot t_{n+2} \cdots t_{n}} + \cdots$$

$$\leq r + \frac{r^2 2^j + r^4 4^j + \cdots + r^2(n-2)j + r^2(n-1)}{1 - r^2 j}.$$

ii) A similar argument as in (i) shows that for any $i = 0, 1, 2, \ldots$,

$$\frac{V^{n_i n + i + j}}{V^{n_i n + i + j + 1}} < \frac{r}{1 - r^2 j}.$$

iii) In the case (i), consider (*) again for $j = 2$.

$$\sum t_{k_1} \cdots t_{k_{n-1}} \sum t_{k_1} \cdots t_{k_{m-1}} = 1 + r^2 + r^4 + \cdots + r^2(n-1).$$

For each $m = 2, \ldots, n - 1$.

From (*) in the proof of (i),

$$\frac{V^{n_0 n + 2}}{V^{n_0 n + 1}} < r + \frac{r^2(1+r^2+\cdots+r^2(n-2))}{1+r^2+\cdots+r^2(n-1)} + \frac{r^4(1+r^2+\cdots+r^2(n-3))}{1+r^2+\cdots+r^2(n-1)} + \cdots$$

$$= \frac{r(1 + r^4 4^1 - r^2(n-1) + r^8 8^1 - r^2(n-2) + \cdots + r^4(n-2) - 1 - r^2 2n + r^4(n-1) - 1 - r^2 2n) + r^4(n-1) - 1 - r^2 2n + r^4(n-1) - 1 - r^2 2n)}{1 - r^2 2n + r^4(n-1) - 1 - r^2 2n + r^4(n-1) - 1 - r^2 2n}.$$

iv) $\frac{V^{n_i n + i}}{V^{n_0 n}} < r^{2i}$. v) From ii) and iv),

$$\frac{V^{n_i n + i + p}}{V^{n_0 n}} < \frac{V^{n_i n + i} V^{n_i n + i + 1} \cdots V^{n_i n + i + p}}{V^{n_0 n} V^{n_i n + i} V^{n_i n + i + p - 1}} < r^{2in}(r R_1)(r R_2) \cdots (r R_p) < r^{2in} r^p R_1 P,$$ where $R_j = 1/(1 - r^2j).$
\[(rR_1)(rR_2) \cdots (rR_p) < (rR_1)^{q-1}(rR_q)^{p-q+1} \]
\[< (rR_1)/(rR_q)^{q-1}(rR_q)^{q-1}(rR_q)^{p-q+1} \]
\[< (1/1 - r^2)^{q-1}(rR_q)^p \]
\[< F_q(rR_q)^p \text{ for } q \leq p, \text{ where } F_q = (1/1 - r^2)^{q-1}. \]

**Notation.** Let \( A \) and \( B \) be two matrices. \( A < B \) means that \( a_{ij} \leq b_{ij} \) for all \( i, j \). Consider \( V^n/V^n_{0n} = \)
\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 \\
V^n_{0n+1} \\
V^n_{0n+2} \\
\vdots \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
V^n_{0n+1} x_1 + V^n_{0n+2} x_2 + V^n_{0n+3} x_3 + \cdots + V^n_{0n+s} x_s \\
V^n_{0n+2} x_1 + V^n_{0n+2} x_2 + V^n_{0n+3} x_3 + \cdots + V^n_{0n+s} x_s \\
\vdots \\
\end{bmatrix}
\]
For \( x = (1, x_1, \cdots)^t \in l^2, \)
\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
V^n_{0n+1} x_1 + V^n_{0n+2} x_2 + V^n_{0n+3} x_3 + \cdots + V^n_{0n+s} x_s \\
V^n_{0n+2} x_1 + V^n_{0n+2} x_2 + V^n_{0n+3} x_3 + \cdots + V^n_{0n+s} x_s \\
\vdots \\
\end{bmatrix}
\]
\[= \sum_{p=0}^{s-1} V^n_{0n+s} x_p \text{ for all } s \geq 0. \]

Let \( A_1 = \)
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & V^1_{02} & V^1_{03} & \cdots \\
0 & V^1_{01} & V^2_{02} & \cdots \\
0 & V^2_{01} & V^2_{02} & \cdots \\
0 & 0 & \cdots \\
\end{bmatrix}
\]
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\[
A_2 = \begin{bmatrix}
0 & x_1 & x_2 & x_3 & \cdots \\
0 & \frac{V_1^{1,2}}{V_{01}} x_1 & \frac{V_1^{1,2}}{V_{01}^2} x_1 + \frac{V_1^{2,2}}{V_{01}^2} x_2 & \cdots \\
0 & \frac{V_2^{1,2}}{V_{02}^2} x_1 & \cdots \\
0 & & & & \ddots \\
0 & & & & & \ddots
\end{bmatrix}
\]

i.e.

\[
(A_1)_{n+n+s} = \begin{cases}
V^n_{0n+s}/V^n_{0n} & n \geq 1, s \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
(A_2)_{n+n+s} = \begin{cases}
x_s & n = 0, s \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( D = \begin{bmatrix}
1 & 0 \\
V_{01}^1 & V_{02}^2 \\
0 & \ddots
\end{bmatrix} \)

Let \( D_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & rR_1 & (rR_1)(rR_2) & (rR_1)(rR_2)(rR_3) & \cdots \\
0 & (rR_1) & (rR_1)(rR_2) & \cdots \\
0 & 0 & (rR_1) & \cdots \\
0 & 0 & 0 & \ddots
\end{bmatrix} \)

\[
(D_1)_{n+n+s} = \begin{cases}
(rR_1)(rR_2) \cdots (rR_s) & n \geq 1, s \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

where \( R_j = 1/(1 - r^{2j}) \).

Let \( y = (1, |x_1|, |x_2|, \cdots) \), i.e. \( y_i = |x_i| \), and suppose that \( \hat{A}_2 \) has the same entries as \( A_2 \), except that \( y_i \) is substituted for \( x_i \) for each \( i \).

Let \( D_2 = \begin{bmatrix}
0 & y_1 & y_2 & y_3 & \cdots \\
0 & r^2 y_1 & r^4 (rR_2) y_1 + r^4 y_2 & \cdots \\
0 & r^4 y_1 & \cdots \\
0 & 0 & \ddots
\end{bmatrix} \)
(D_2)_{n+n+s} = \begin{cases} 
  y_s & n = 0, s = 0 \\
  0 & \text{otherwise} 
\end{cases}

Lemma 3.4. Assume \( t_{n+1}/t_n \leq r < 1 \) for all \( n \). Then \( s_x(V^*) = D(I + A_1 + A_2), \) \( A_1 < D_1, \) and \( A_2 < D_2. \) Moreover, \( A_1 + \hat{A}_2 \) is less than

\[
\begin{bmatrix}
  0 & y_1 & y_2 & y_3 & \cdots \\
  0 & rR_1 + r^2y_1 & (rR_1)(rR_2) + r^2(rR_1)y_1 + r^4y_2 & \cdots \\
  0 & rR_1 + r^4y_1 & \cdots \\
  0 & 0 & \cdots 
\end{bmatrix}
\]

Proof. \( S_x(V^*) = [x, V^*x, V^{*2}x, \ldots]^t \)

\( = D[x, \frac{V^*x}{V^*1^0_1}, \frac{V^{*2}x}{V^*2^0_2}, \ldots]^t. \)

And \( [x, \frac{V^*x}{V^*1^0_1}, \frac{V^{*2}x}{V^*2^0_2}, \ldots]^t = I + A_1 + A_2 \) from the above computation and the definitions of \( A_1 \) and \( A_2. \) Hence \( S_x(V^*) = D(I + A_1 + A_2). \) From Lemma 3.3 (1) and the above definitions of \( A_2, D_1, \) and \( D_2, A_1 < D_1 \) and \( \hat{A}_2 < D_2. \) Moreover \( A_1 + \hat{A}_2 < D_1 + D_2 \)

\[
\begin{bmatrix}
  0 & y_1 & y_2 & y_3 & \cdots \\
  0 & rR_1 + r^2y_1 & (rR_1)(rR_2) + r^2(rR_1)y_1 + r^4y_2 & \cdots \\
  0 & rR_1 + r^4y_1 & \cdots \\
  0 & 0 & \cdots 
\end{bmatrix}
\]

Let

\[
C_0 = \begin{bmatrix}
  0 & rR_1 + r^2y_1 & (rR_1)(rR_2) + r^2(rR_1)y_1 + r^4y_2 & \cdots \\
  0 & rR_1 + r^4y_1 & \cdots \\
  0 & rR_1 + r^6y_1 & \cdots \\
\end{bmatrix}
\]

Let
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C_1 = \begin{bmatrix} 0 & rR_1 & (rR_1)(rR_2) & (rR_1)(rR_2)(rR_3) & \cdots \\ 0 & rR_1 & (rR_1)(rR_2) & \cdots \\ 0 & rR_1 & \cdots \\ \end{bmatrix}

be a Toeplitz operator induced by \( \varphi = \sum_{n=2}^{\infty} \pi_{k=1}^{n-i}(rR_k)e_n \), [5,p.136], where \( R_k = 1/(1 - r^{2k}) \). Now \( 0 < r < 1 \). Choose \( g \) large enough such that \( r^2R_2 \leq 1 \). Let

\[
C_2 = F_q \begin{bmatrix} 0 & r^2y_1 & r^2(R) + r^4y_2 & r^2(R)^2 + r^4(rR)y_2 + r^6y_3 & \cdots \\ 0 & r^4y_1 & r^4(rR)y_2 + r^6y_2 & \cdots \\ 0 & r^6y_1 & \cdots \\ 0 & \cdots \\ \end{bmatrix}
\]

where \( R = R_q = 1/1 - r^2q \). Then \( C_0 < C_1 + C_2 \).

Lemma 3.5. If \( 0 < r < 1 \) and \( q \) is chosen so that \( r^2R_q \leq 1 \), then \( C_2 \) is a Hilbert-Schmidt operator.

Proof. We will let \( R = R_q, q \leq p \). Let \( C_3 \) be the matrix whose \( q - 1 \) super-diagonals are zero and the other entries are the same as \( C_2 \). Then for \( j = q > 1 \),

\[
(C_3)_{n,n+j} = (C_2)_{n,n+j} = r^{2n}(rR)^jy_1 + r^{2(2n)}(rR)^{j-1}y_2 + \cdots + r^{j(2n)}(rR)y_j + r^{j+1}(rR)^{j-1}y_{j+1}
\]

for \( j = q, q + 1, \ldots \).

\[
\sum_{j=q}^{\infty} |(C_3)_{n,n+j}|^2 = \sum_{j=q}^{\infty} |(C_2)_{n,n+j}|^2 
\leq \sum_{j=q}^{\infty} r^{2(2n+j-1)}(\sum_{s=1}^{j+1} R_j - s + 1 + r^{2(n-1)}(s-1)y_s)^2
\leq r^{4n} \sum_{j=q}^{\infty} r^{2(j-1)}[\sum_{s=1}^{j+1} r^{2s} R^2(j-s+1) + r^4(n-1)(s-1)](\sum_{j=1}^{j+1} y_s)^2
\leq Mr^{4n} \sum_{j=q+1}^{\infty} r^{2(j-2)}[\sum_{s=1}^{j+1} r^{2s} R^2(j-s+1)],
\]
where \( \sum_{s=0}^{\infty} y_s^2 = M < \infty \)
\[
\leq M r^{4n} r^{-4} \sum_{j=q+1}^{\infty} r^{2j} (\sum_{s=1}^{j} r^{2s} R^2(j-s+1)) = M T r^4(n^j-1),
\]
where \( T \)
\[
= \sum_{j=q+1}^{\infty} r^{2j} (\sum_{s=1}^{j} r^{2s} R^2(j-s+1)) < \infty
\]
since, by the choice of \( q, r^2 R^2 < 1 \) (and of course \( r^2 / R^2 < 1 \)). Thus
\[
\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |(C_3)_{n+n+1}|^2 \leq \sum_{n=1}^{\infty} M T r^4(n-1) = M T / (1 - r^4) < \infty.
\]
Clearly, \( C_2 - C_3 \) is a finite sum of \( q - 1 \) super-diagonal Hilbert-Schmidt operators. Thus, \( C_2 \) is a Hilbert-Schmidt operator.

Lemma 3.6. If \( r \leq 0.43 \), then \( I + C_1 \) is invertible.

Proof.
\[
C_1 = \begin{bmatrix}
0 & r R_1 & (r R_1)(r R_2) & (r R_1)(r R_2)(r R_3) & \cdots \\
0 & r R_1 & (r R_1)(r R_2) & \cdots \\
0 & r R_1 & \cdots \\
0 & \cdots 
\end{bmatrix}
\]
is the Toeplitz operator \( T_\varphi \) induced by \( \varphi \), where \( \varphi \) is a bounded measurable function on the unit circle \( \Gamma \) with the Fourier expansion \( \sum_{n=2}^{\infty} (\pi_{k=1}^{n-1} r R_k) e_n \).
It is known that \( ||T_\varphi|| \leq ||\varphi||_\infty = \inf\{M : m(\Gamma) = 0\} \), [5, p.138]. But
\[
||\varphi||_\infty = \inf\left\{M : m(\Gamma) = 0, \varphi(\Gamma) > M\right\} = 0
\]
for all \( k = 3, 4, \ldots \)
\[
|\varphi(z)| = |\sum_{n=2}^{\infty} (\pi_{k=1}^{n-1} r R_k) z^n| \\
\leq \sum_{n=2}^{\infty} (\pi_{k=1}^{n-1} r R_k) |z|^n \\
= \sum_{n=2}^{\infty} (\pi_{k=1}^{n-1} r R_k) \\
< r R_1 + \sum_{n=1}^{\infty} (r R_1)(r R_2)^2, \text{ since } R_k < R_2
\]
for all \( k = 3, 4, \ldots \)
\[
= r R_1 + (r R_1)(r R_2)[1 + r R_1 + (r R_2)^2 + \cdots] \\
= r R_1 + (r R_1)(r R_2)[1/(1 - r R_2)] \\
= (r + r^3)/(1 - r - r^4), \text{ since } R_j = 1/(1 - r^2j).
\]
We can easily show that \( (r + r^3)/(1 - r - r^4) < 1 \), if \( 0 < r \leq 0.43 \), i.e., \( m\{\Gamma : |\varphi(z)| > 1\} = 0 \) if \( r \leq 0.43 \). So, \( ||T_\varphi|| \leq ||\varphi||_\infty < 1 \). Hence \( I + T_\varphi = I + C_1 \) is invertible.

Theorem 3.7. If \( t_n + 1/t_n \leq r \leq 0.43 \), then \( I + A_1 \) is invertible and \( A_2 \) is compact.
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Proof. By Lemma 3.5, $C_2$ is a Hilbert-Schmidt operator. Then $D_2$ is Hilbert-Schmidt, since $x \in l^2$. Hence $A_2$ is Hilbert-Schmidt. So $A_2$ is Hilbert-Schmidt. From the proof of Lemma 3.6, $\|C_1\| < 1$. But $\|D_1\| \leq \|C_1\|$ and $A_1 < D_1$. Let $A_1 = (a_{ij})$ and $D_1 = (d_{ij})$. For $x = (1, x_1, \ldots) \in l^2$, $y = (1, |x_1|, \ldots)$. So

$$\|A_1y\|^2 = \sum_{i=0}^{\infty} (\sum_{j=i+1}^{\infty} a_{ij}y_j)^2 \leq \sum_{i=0}^{\infty} (\sum_{j=i+1}^{\infty} |a_{ij}|y_j)^2 \leq \sum_{i=0}^{\infty} (\sum_{j=i+1}^{\infty} d_{ij}y_j)^2 = \|D_1y\|^2 \leq \|D_1\|^2 \|y\|^2 < \|y\|^2$$

So, $\|A_1\| < 1$. Hence $I + A_1$ is invertible.

Theorem 3.8. If $t_n + 1/t_n \leq 0.43$, then the adjoint $V^*$ of the discrete Volterra operator $V$ is unicellular.

Proof. From Theorem 3.7 and 2.4, $I + A_1 + A_2$ is one-to-one. But $S_x(V^*) = D(I + A_1 + A_2)$ from Lemma 3.4, and $D$ is one-to-one. Hence $S_x(V^*)$ is one-to-one. For a nonnegative integer $N \geq 1$,

$$U^N V^* U^N = \begin{bmatrix}
0 \\
t_N t_{N+1} \\
t_N t_{N+2} & t_{N+1} t_{N+2} \\
t_N t_{N+3} & t_{N+1} t_{N+3} & t_{N+2} t_{N+3} & 0 \\
\vdots
\end{bmatrix}$$

Since $U^N V^* U^N$ has the same hypothesis in $V^*$, $S_x(U^N V^* U^N)$ is one-to-one for any nonnegative integer $N \geq 1$. By Theorem 2.3, $V^*$ unicellular.

References


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