J.S. BIRMAN'S CONJECTURES ON BRAIDS AND LINKS

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Dedicated to Professor Younki Chae on his sixtieth birthday

The purpose of this paper is to outline the relationship between braids and links and to survey J. S. Birman's conjectures on split braids and composite links. It is proved that there are infinitely many braids which are not conjugate to split (separate) braids but whose closures are composite (split) links for \( n \geq 6 \) respectively and there are infinitely many pairs of non-conjugate braids having same isotopic closures for \( n \geq 4 \) and every composite (split) link is isotopic to a closure of a split (separate respectively) braid.

1. Introduction

A knot is a simple closed polygonal curve in \( E^3 \) and a link is a mutually disjoint union of knots. The classical braid group \( \pi_1 B_{0,n} E^2 \) admits a presentation with generators \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \) and defining relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) if \( |i - j| \geq 2 \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for \( 1 \leq i \leq n - 2 \). For a braid \( \beta \in B_n \) let \( \overline{\beta} \) denote a link obtained by closing a geometric braid representing \( \beta \). \( \overline{\beta} \) is called a closure of \( \beta \). Notice that \( \overline{\beta} \) is unique up to isotopy. Consider a map \( \phi \) from the set of all braids to the set of all links defined by \( \phi(\beta) = \overline{\beta} \). Then \( \phi \) is well defined and it is due to J. Alexander that the map \( \phi \) is surjective. The braid index of \( L \) is the smallest integer \( n \) such that \( \overline{\beta} = L \) with \( \beta \in B_n \). Note that conjugate braids have same

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isotopic closures. But there exist non-conjugate braids which have same isotopic closures. We use the notation $(\beta, n)$ when $\beta$ is a braid in $B_n$. The following theorem is due to A. Markov.

**Theorem.** (Markov) Let $\bar{\beta}$ and $\bar{\beta}^*$ be two closed braids in $E^3$, with braid representatives $(\beta, n)$ and $(\beta^*, n^*)$. Then $\bar{\beta}$ is isotopic to $\bar{\beta}^*$ if and only if there is a finite sequence of moves $(\beta, n) = (\beta_0, n_0) \to \cdots \to (\beta_k, n_k) = (\beta^*, n^*)$ joining $(\beta, n)$ to $(\beta^*, n^*)$, such that for each $0 \leq i < k$, the braid $(\beta_{i+1}, n_{i+1})$ can be obtained from its predecessor $(\beta_i, n_i)$ by applying one of the following moves:

$W_1$: Replace $\beta_i$ by any other braid in $B_{n_i}$, which is conjugate to $\beta_i$ and set $n_{i+1} = n_i$.

$W_2$: Replace $(\beta_i, n_i)$ by $(\beta_i, \sigma_1^{\pm 1}, n_i + 1)$ or, if $\beta_i = \gamma \sigma_{n_i}^{\pm 1}$ where the braid word $\gamma$ involves the generators $\sigma_1, \ldots, \sigma_{n_i-2}$ only, replace $(\beta_i, n_i)$ by $(\gamma, n_i - 1)$.

The moves $W_1$ and $W_2$ are called the Markov movements.

Now, we recall definitions of some special types of links and braids. A knot $K$ is called a composite knot if it is a connected sum of two non-trivial knots. Otherwise it is called a prime knot. Simultaneously we can define a composite link and a prime link. A link $L$ is called a split link if there is a 2-sphere $X$ in $S^3 \setminus L$ such that $X$ does not bound a closed 3-ball. Equivalently, we redefine a composite link by saying that there is a 2-sphere $X$ in $S^3$ which meets the link in 2 points and decompose it into sublinks $K_1 \# K_2$ neither of which is unknotted arcs. A braid $\beta \in B_n$ is called a split braid if $\beta$ can be written as $\beta = \beta_1 \beta_2$ where $\beta_1$ is a word in $\sigma_1, \sigma_2, \ldots, \sigma_k$ and $\beta_2$ is a word in $\sigma_{k+1}, \ldots, \sigma_{n-1}$ for some $1 < k < n - 1$. A braid $\beta \in B_n$ is called a separate braid if it can be represented by $\beta_1 \beta_2$ where $\beta_1$ is a word in $\sigma_1, \ldots, \sigma_k$ and $\beta_2$ is a word in $\sigma_{k+2}, \ldots, \sigma_{n-1}$, for some $1 < k < n - 2$. Geometrically, it is clear that a closure of a split braid is a composite link and that a closure of a separate braid is a split link.

J. S. Birman raised some conjectures on the relationship between braids and links.

1) For $\beta \in B_n$, $\beta$ is prime if and only if $\beta$ is not conjugate to a split braid.

2) For two braids $\beta_1, \beta_2 \in B_n$, $\beta_1$ is conjugate to $\beta_2$ if and only if $\overline{\beta_1}$ is isotopic to $\overline{\beta_2}$.

3) A knot $K$ is composite if and only if it is isotopic to a closure of a split braid.

Recently, H. R. Morton[5] published an important theorem for the
conjecture 2). For a fixed braid axis $A$, he proved that $\beta_1$ is conjugate
to $\beta_2$ if and only if $\overline{\beta_1}$ is isotopic to $\overline{\beta_2}$ in $S^3 \setminus A$. In [4], J. S. Birman
and W. Menasco solved the conjectures 1) and 2) by introducing a new
link operation called an exchange move. We proved the conjecture 2)
independently but it was turned out to be covered by their split and
composite braid theorem which read:

**Theorem (Birman and Menasco [4])** Let $L$ be a split (composite) link
and let $L$ be an arbitrary closed $n$-braid representative of $L$. Then there
exist a split (composite, respectively) $n$-braid $K^*$ which represents $L$ and a
finite sequence of closed braids: $K = K_0 \rightarrow K_1 \rightarrow K_2 \rightarrow \cdots \rightarrow K_m = K^*$
such that each $K_{i+1}$ is obtained from $K_i$ by either isotopy in $S^3 \setminus A$ or an
exchange move.

The purpose of this paper is to show the following.

1) There are infinitely many braids which are not conjugate to split
(singlet) braids but whose closures are composite (split) links for $n \geq 6$ (4
respectively).

2) There are infinitely many pairs of non-conjugate braids having same
isotopic closures for $n \geq 4$.

3) Every composite (split) link is isotopic to a closure of a split (sepa-
rate respectively) braid.

2. J. S. Birman’s conjectures

In this section, we describe Birman’s conjectures on split braids and
composite knots together with some known results up to date.

The following conjectures on the relationship between split braids and
composite knots were raised by J. S. Birman.

1) For $\beta \in B_n$, $\beta$ is prime if and only if $\beta$ is not conjugate to a split
braid. Or equivalently, $\beta$ is a composite link if and only if $\beta$ is conjugate
to a split braid [p.99, 1].

2) For two braids $\beta_1, \beta_2 \in B_n$, $\beta_1$ is conjugate to $\beta_2$ if and only if $\overline{\beta_1}$ is
isotopic to $\overline{\beta_2}$ [2].

3) A link $L$ is prime if and only if it is not isotopic to a closure of a split
braid. Or equivalently, a link $L$ is composite if and only if it is isotopic to
a closure of a split braid [p.217, 1].

For the conjecture 1), if $\beta$ is prime, then clearly $\beta$ is not conjugate to
a split braid. For the converse, it is true when $n = 2$, and H. R. Morton
[6] provided a partial answer to the converse. It is true for $n = 3$ and
he found a counterexample for the converse when \( n = 5 \). Recently J. S. Birman and W. W. Menasco provided an answer for \( n = 4 \) [4]. In example 3.1, we construct infinitely many braids which are not conjugate to split braids but whose closures are composite links for \( n \geq 6 \).

For the conjecture 2), if \( \beta_1 \) is conjugate to \( \beta_2 \) then clearly \( \overline{\beta_1} \) is isotopic to \( \overline{\beta_2} \). For the converse, by applying Markov movements of type \( W_2 \) we can easily see that there are non-conjugate braids having same isotopic closures for each \( n \geq 2 \). J. S. Birman restated the conjecture 2) by additionally requiring that \( \beta_1 \) and \( \beta_2 \) have braid index \( n \). For the converse, it is true for \( n = 2 \) if \( \beta_1, \beta_2 \) have at least three crossings. K. Murasugi and R. S. D. Thomas [7] constructed counter examples for \( n = 3 \) and 4. In the case of \( n = 4 \), they took \( \alpha_1 = \sigma_1^m \sigma_2^p \sigma_3^q \alpha_2 = \sigma_1^m \sigma_2^q \sigma_3^p \) with \( m, n, p \) distinct, odd, and at least three in absolute value. Then the two closed braids \( \overline{\alpha_1}, \overline{\alpha_2} \) disregarding the choice of orientation around the axis, are not isotopic in \( S^3 \setminus \text{the braid axis} \). But in the case of \( n = 3 \), they took \( \beta_1 = \sigma_1^{-2} \sigma_2^{-2} \sigma_3 \beta_2 = \sigma_1 \sigma_2^{-2} \sigma_2^{-2} \sigma_3^{-1} \) then two closed braids \( \overline{\beta_1}, \overline{\beta_2} \) disregarding the choice of orientation around the axis, are equivalent in \( S^3 \). So it is still open whether \( \beta_1 \) and \( \beta_2 \) belong to the same equivalent class which is defined by conjugation and respecting the choice of orientation around the axis provided that \( \overline{\beta_1} \) is isotopic to \( \overline{\beta_2} \) in the case of \( n = 3 \). In example 3.2 we construct infinitely many non-conjugate pairs of braids having same isotopic closures for \( n \geq 4 \) respecting the choice of orientation around axis and having braid index \( n \).

Related to the conjecture 2), J. S. Birman asked how to determine an essential condition in the following statement. For two braids \( \beta_1, \beta_2 \in B_n \) with braid index \( n \) and with some condition, \( \beta_1 \) is conjugate to \( \beta_2 \) if and only if \( \beta_1 \) is isotopic to \( \overline{\beta_2} \) [1]. This question was a starting point of a series of the papers on studying links via braid groups. When \( n = 3 \), if \( \beta_1 \) is a composite link or a split link or a torus link then the converse is true without any extra condition.

For the conjecture 3), if \( K \) is isotopic to a closure of a split braid then it is composite. For the converse, it is true and we proved it in Theorem 3.3 independently but it turns out to be a weaker form of the composite theorem [4].

Next we consider relationships between split links and separate braids. We ask ourselves the following questions and provide complete answers to them.

4) For \( \beta \in B_n \), \( \overline{\beta} \) is a split link if and only if \( \beta \) is conjugate to a separate braid.
5) A link $L$ is a split link if and only if it is isotopic to a closure of a separate braid.

For the question 4), If $\beta$ is conjugate to a separate braid, then $\overline{\beta}$ is a split link. For the converse, it is true for $n = 2$ and 3. We construct braids which are not conjugate to separate braids but whose closures are split links for $n \geq 4$ in example 3.2.

For the question 5), If $L$ is isotopic to a closure of a separate braid then it is a split link. We can show that every split link is a closure of a separate braid by using a similar method as in the proof of theorem 3.3.

3. Main results

Consider, homomorphisms $\phi_n : B_n \to B_5$ and $\phi'_n : B_n \to B_3$ defined by $\phi_n(\sigma_i) = \sigma_i$ for $i = 1, 2, 3$, $\phi_n(\sigma_j) = 1$ for $j = 4, \ldots, n-2$ and $\phi_n(\sigma_{n-1}) = \sigma_4$ and $\phi'_n(\sigma_i) = \sigma_i$ for $i = 1, 2$, $\phi'_n(\sigma_j) = 1$ for $j = 3, \ldots, n - 1$.

As explained in section 2, for the converse of the conjecture 1), we get the following counter examples.

Example 3.1. In Figure 1, $C, D, C_4, C_5, \ldots, C_{n-2}$ are braids whose closures are prime knots with 3, 3, 4, 5, $\ldots, n - 2$ strings respectively and $C, D, \phi'_4(C_4), \phi'_5(C_5), \ldots, \phi'_{n-2}(C_{n-2})$ are braids whose closures are prime
knots with braid index 3. For instance, \( C_4 = (\sigma_1^{-1}\sigma_2)^2\sigma_4 \). \( a \) is the H. R. Morton's example for \( n = 5 \). We extend his example for \( n \geq 6 \) in \( b \).

**Proof.** We explain the examples when \( n = 6 \). The other cases can be explained similarly. Here, we denote a braid with \( n \)-string in the example by \( \beta_n \).

Suppose \( \beta_6 \) is conjugate to a split braid. There are braids \( \omega \) and \( \gamma \) in \( B_6 \) such that \( \gamma \) is a split braid and \( \omega \beta_6 \omega^{-1} = \gamma = \gamma_1(\sigma_1, \sigma_2) \gamma_2(\sigma_3, \sigma_4, \sigma_5) \). Since \( \phi_6(\sigma_i) = \sigma_i \) for \( i = 1, 2, 3 \), \( \phi_6(\sigma_4) = 1 \), \( \phi_6(\sigma_5) = \sigma_4 \). Then, there is a braid \( \phi_6(\omega) \) in \( B_5 \) such that \( \phi_6(\omega) \phi_6(\beta_6) \phi_6(\omega)^{-1} \) is the split braid \( \phi_6(\gamma) \). This is a contradiction to the construction of H. R. Morton for \( n = 5 \).

**Example 3.2.** Let \( \beta_4 = (\sigma_1)^3\sigma_2(\sigma_3)^3\sigma_2^{-1}, \beta_5 = (\sigma_1)^3\sigma_2(\sigma_3\sigma_4^{-1})^2\sigma_2^{-1}, \ldots, \beta_n = (\sigma_1)^3\sigma_2 \phi_{n-2}''(C_{n-2}) \sigma_2^{-1} \), where \( C_{n-2} \) is a braid in \( B_{n-2} \) with braid index \( n - 2 \) and \( \phi_{n-2}'' \) is a homomorphism from \( B_{n-2} \) to \( B_n \) defined by \( \phi_{n-2}''(\sigma_i) = \sigma_{i+2} \). Then \( \beta_n \) is not conjugate to the corresponding separate braid \( \sigma_1 \phi_{n-2}'' \) but \( \overline{\beta_n} \) is a split link and \( \beta_n \) and \( \sigma_1 \phi_{n-2}'' \) have the same braid index \( n \). So, there are infinitely many pairs of non-conjugate braids having same isotopic closures with braid index \( n \) for \( n \geq 4 \) respecting the choice of orientation around the braid axis.

**Theorem 3.3.** A link \( L \) is composite if and only if it can be represented by a split braid.

Before proving the theorem we are going to describe the basic deformations following J. S. Birman [1]. Let \( L \) be a link with edge \( [a, c] \), and let \( b \) be a point which is not on \( L \). Suppose that \( [a] \cap [b, c] = [a, b] \cap [c] = \emptyset \) and \( [a, b, c] \cap L = [a, c] \), where \( [a, b, \ldots, z] \) denotes the convex hull of the points \( a, b, \ldots, z \). We can define an operation \( \mathcal{E}_{ac}^b \) by \( \mathcal{E}_{ac}^b L = L - [a, c] + [a, b] + [b, c] \). The operation \( \mathcal{E}_{ac}^b \) and its inverse are called type \( \mathcal{E} \)-deformations. Two links \( L \) and \( L^* \) are said to be combinatorially equivalent if there exists a finite sequence of links joining \( L \) to \( L^* \) such that each link in the sequence can be obtained from its predecessor by a single type \( \mathcal{E} \)-deformation. Notice that combinatorial equivalence and isotopic equivalence are the same on tamed links and we use combinatorial equivalence. We need the following lemmas to prove theorem 3.3.

**Lemma 3.4.** Let \( L \) be a link with a fixed edge. Then \( L \) is combinatorially equivalent to a closure of a braid leaving the fixed edge invariant under \( \mathcal{E} \)-deformations in the sequence.

**Proof.** Given a fixed edge \( e = [a, b] \) of a link \( L \), we may assume that \( [a, b] \) is
always positive. Otherwise we can change the orientation of the rotation so that \([a, b]\) is positive. If \([a, b] > 0\), then it is routine to check that \([a, b]\) is invariant under the equivalence constructed in the J. S. Birman’s original proof of the theorem 2.1 in [p.42,1].

**Lemma 3.5.** Given a closed braid \(\overline{\beta}\), there is a braid \(\gamma\) in \(B_n\) such that \(\beta = \gamma\) and an edge of \(\overline{\beta}\) occurs at the last string of \(\overline{\gamma}\).

**Proof.** Using a simple string isotopy it follows easily as Figure 2 illustrates.

![Figure 2](image)

**Proof of theorem 3.3.** Suppose that a link \(L\) is composite. Then there are nontrivial links \(L_1, L_2\) such that \(L = L_1 \# L_2\) and there are two inessential imbeddings of 2-spheres \(X_1\) which bound \(L_i\) for \(i = 1, 2\). Let \(e_1, e_2\) be two edges in \(L_1, L_2\) respectively which are in the connection. By lemma 3.4, we may assume that every operation in the equivalence of \(L_1 \sim \overline{\alpha}\) avoids \(X_2\) and leaves the connecting edge \(e_1\) invariant. Similarly, there is a braid \(\beta \in B_n\) such that \(L_2 \sim \overline{\beta}\) and every link operation in the equivalence avoids \(X_1\) and the connecting edge \(e_2\) is invariant under the equivalence. By lemma 3.5, there are two braids \(\mu \in B_n, \nu \in B_m\) such that \(e_1, e_2\) occur at the last string of \(\overline{\mu}, \overline{\nu}\) respectively, and \(\overline{\mu} \sim \overline{\alpha}, \overline{\nu} \sim \overline{\beta}\). Since \(X_1\) and \(X_2\) are avoided under equivalences, \(L_1 \# L_2\) is isotopic to \(\overline{\mu} \# \overline{\nu}\). So we get a form of \(\overline{\mu} \# \overline{\nu}\) as in the first picture of the figure 3.

Now we only need to show that there is \(\gamma \in B_{n+m-1}\) such that \(\gamma\) is a
split braid and $\gamma \sim \mu \# \nu$ and the figure 3 which is a sequence of string isotopies, shows how to choose such a braid $\gamma$.

![Figure 3](image)

**Figure 3**

**Remark 3.6.** We can prove that every split link can be represented by a separate braid by using a similar method as in the proof of theorem 3.3.

**References**


On J.S. Birman's conjecture on braids and links


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