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THE COMPARISON OF d-MEASURE WITH PACKING AND HAUSDORFF MEASURES

Hung Hwan Lee* and In Soo Baek**

Dedicated to Professor Younki Chae on his sixtieth birthday

1. Introduction and definitions

The idea of dimension is fundamental in the study of fractals [2]. Various definitions of dimension have been proposed, such as the Hausdorff dimension, the packing dimension and the modified lower and upper box dimensions etc. Unlike the Hausdorff and packing dimensions the modified lower and upper box dimensions are not defined in terms of measures. However it is well-known that the modified upper box dimension is just the same as the packing dimension [2]. Accordingly we [4] recently have defined a new measure called the d-measure which determines the modified lower box dimension. Together with several properties of this d-measure, we showed that the length of a rectifiable curve equals to its 1-dimensional Hausdorff, packing measures and d-measure.

In this note we first show that 1-dimensional packing, Hausdorff, Lebesgue measures and d-measure coincide on $\mathbf{R}$. We also compare the s-dimensional d-measure with the s-dimensional packing and Hausdorff measures on $\mathbf{R}^n$. Finally we show that the 1-dimensional Hausdorff, packing and d-measures have same value for p-regular sets in $\mathbf{R}^n$.

We begin by introducing the necessary definitions.

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For $0 \leq s < \infty$ the s-dimensional Hausdorff outer measure of $E \subset \mathbf{R}^n$ is given, in the usual way, by

$$H^s(E) = \lim_{\delta \to 0} \inf\{\sum_i |U_i|^s : E \subset \cup_{i=1}^{\infty} U_i, 0 < |U_i| < \delta\}$$

where $|\ |$ denotes the diameter of a set. A δ-packing of E is a family of pairwise disjoint balls with centers in E and diameters less than or equal to δ. The s-dimensional packing premeasure of E is $P^s(E) = \lim_{\delta \to 0} \sup\{\sum_i |U_i|^s : \{U_i\}_{i=1}^{\infty}$; a δ-packing of $E\}$. The s-dimensional packing outer measure of E is

$$p^s(E) = \inf\{\sum_{n=1}^{\infty} P^s(E_n) : E \subset \cup_{n=1}^{\infty} E_n\}.$$

Let $N(E, \delta)$ denote the smallest number of closed balls of diameter δ that cover E. The s-dimensional d-premeasure [4] of a bounded set E in $\mathbf{R}^n$ is

$$D^s(E) = \liminf_{\delta \to 0} N(E, \delta)\delta^s.$$

The s-dimensional d-measure [4] of $E \subset \mathbf{R}^n$ is

$$d^s(E) = \inf\{\sum_{n=1}^{\infty} D^s(E_n) : E \subset \cup_{n=1}^{\infty} E_n, E_n \text{ are bounded}\}.$$

The Hausdorff dimension, packing dimension (modified upper box dimension), and modified lower box dimension can be defined by

$$\begin{aligned}
\dim_H(E) &= \inf\{s \geq 0 : H^s(E) = 0\} = \sup\{s \geq 0 : H^s(E) > 0\} \\
\dim_p(E) &= \inf\{s \geq 0 : p^s(E) = 0\} = \sup\{s \geq 0 : p^s(E) > 0\} \\
\underline{\dim}_{MB}(E) &= \inf\{s \geq 0 : d^s(E) = 0\} = \sup\{s \geq 0 : d^s(E) > 0\}
\end{aligned}$$

Also we define a rarefaction index, the lower box dimension of a bounded set E in $\mathbf{R}^n$ by

$$\underline{\dim}_B(E) = \inf\{s \geq 0 : D^s(E) = 0\} = \sup\{s \geq 0 : D^s(E) > 0\}.$$

Clearly $\underline{\dim}_B$ is monotone, but not σ-stable in the sense of [6]. As in [6], $\widehat{\underline{\dim}}_B$ is defined for any $E \subset \mathbf{R}^n$ by

$$\widehat{\underline{\dim}}_B(E) = \inf\{\sup_n \underline{\dim}_B(E_n) : E = \cup_{n=1}^{\infty} E_n, E_n \text{ are bounded in } \mathbf{R}^n\}.$$

Then $\widehat{\dim}_B$ is σ-stable.

If $0 < H^1(E) < \infty$, then E is called a 1-set. We define the lower and upper densities of a 1-set E at a point $x \in \mathbf{R}^n$ as

$$\underline{D}^1(E,x) = \liminf_{r\to 0} \frac{H^1(E \cap B_r(x))}{2r}$$

and

$$\overline{D}^1(E,x) = \limsup_{r\to 0} \frac{H^1(E \cap B_r(x))}{2r}$$

respectively, where $B_r(x)$ is a ball of radius r with center x. For a 1-set E, a point x at which $\underline{D}^1(E,x) = \overline{D}^1(E,x) = 1$ is called a regular point of E, otherwise x is an irregular point. A 1-set is termed regular if H^1-almost all of its points are regular, and irregular if H^1-almost all of its points are irregular.

We also define p-regularity and d-regularity for packing measure and d-measure respectively.

We write $\underline{\triangle}^1(E,x)$, $\overline{\triangle}^1(E,x)$ and $\underline{d}^1(E,x)$, $\overline{d}^1(E,x)$ for the lower and upper densities of a 1-set E at $x \in \mathbf{R}^n$ for packing measure and d-measure respectively. If $\lim_{r\to 0} p^1(E \cap B_r(x))/2r = 1$ for 1-dimensional packing measure almost all $x \in E$, then the packing 1-set E is called a p-regular set. Likewise a d-regular set can be defined.

2. Results

Before we discuss our main results, we review some properties of d-measure [4].

Proposition A. *d^s is a Borel regular. That is, there exists a Borel set $B \subset \mathbf{R}^n$ such that $E \subset B$ and $d^s(E) = d^s(B)$ for any set $E \subset \mathbf{R}^n$.*

Proof. Since $N(F,r) = N(\overline{F},r), D^s(F) = D^s(\overline{F})$. Hence $d^s(E) = \inf\{\sum_{n=1}^{\infty} D^s(\overline{E_n}) : E \subset \cup_{n=1}^{\infty} E_n,\ E_n$ are bounded sets in $\mathbf{R}^n\}$. For any integer $n \geq 1$, there exists a sequence of bounded sets $\{E_{n,i}\}_{i=1}^{\infty}$ such that $E \subset \cup_{i=1}^{\infty} E_{n,i}$ and $\sum_{i=1}^{\infty} D^s(E_{n,i}) < d^s(E) + \frac{1}{n}$. Let $B = \cap_{n=1}^{\infty} \cup_{i=1}^{\infty} \overline{E}_{n,i}$. Then $E \subset B$, and $B \subset \cup_{i=1}^{\infty} \overline{E}_{n,i}$ for every n. Therefore $d^s(B) \leq \sum_{i=1}^{\infty} D^s(\overline{E}_{n,i})$ for every n. Hence $d^s(B) \leq \inf_n \sum_{i=1}^{\infty} D^s(E_{n,i}) \leq d^s(E)$. But, $d^s(E) \leq d^s(B)$.

Proposition B. *d^s is a metric outer measure.*

Proof. Plainly we have $d^s(A \cup B) \leq d^s(A) + d^s(B)$ by subadditivity of d^s. Suppose that Dist $(E,F) > 0$ for bounded sets E, F. Then Dist

$(E,F) > 2\varepsilon > 0$ for some positive constant ε. Thus $N(E\cup F,\varepsilon) = N(E,\varepsilon)+N(F,\varepsilon)$. Hence $D^s(E\cup F)\geq D^s(E)+D^s(F)$. Hence, for A, B such that $\mathrm{Dist}(A,B)>0$,

$$\begin{aligned} d^s(A\cup B) &= \inf\{\sum D^s(E_n) : A\cup B = \cup E_n, E_n \text{ are bounded }\} \\ &\geq \inf\{\sum D^s(E_n\cap A)+\sum D^s(E_n\cap B) : A\cup B=\cup E_n, \\ &\qquad E_n \text{ are bounded }\} \\ &\geq \inf\{\sum D^s(E_n\cap A) : A\cup B=\cup E_n, E_n \text{ are bounded }\} \\ &\quad +\inf\{\sum D^s(E_n\cap B) : A\cup B=\cup E_n, E_n \text{ are bounded }\} \\ &\geq d^s(A)+d^s(B) \end{aligned}$$

Proposition C. *d^s is a regular outer measure.*

Proof. It is immediate from Propositions A and B since the family of all measurable sets of a metric outer measure contains all Borel sets.

Proposition D. *$D^1(\Gamma)=\ell(\Gamma)$ for a curve Γ, where $\ell(\Gamma)$ is the length of Γ.*

Proof. By definition, $H^1(\Gamma)\leq D^1(\Gamma)$. Since $H^1(\Gamma)=\ell(\Gamma)$, $\ell(\Gamma)\leq D^1(\Gamma)$. Suppose that $\ell(\Gamma)<\infty$. Then we can dissect Γ into n-parts of same arc length; $\Gamma_1, \Gamma_2, \cdots$, and Γ_n. Let x_i be the mid-point of Γ_i for length for each $i=1,2,\cdots,n$. Then each Γ_i is contained in the closed ball of center x_i with radius $\ell(\Gamma)/2n$. Thus $N(\Gamma,\ell(\Gamma)/n)\ell(\Gamma)/n\leq n(\ell(\Gamma)/n)=\ell(\Gamma)$. Therefore $\liminf_{n\to\infty} N(\Gamma,\ell(\Gamma)/n)\ell(\Gamma)/n\leq\ell(\Gamma)$. Hence $D^1(\Gamma)\leq\ell(\Gamma)$.

Let $\mathcal{L}^1$ denote the 1-dimensional Lebesgue measure. It is easy to see that H^1, p^1, and $\mathcal{L}^1$ have same value for any subset of $\mathbf{R}$. It is natutal to ask whether the definitions of d^1 and $\mathcal{L}^1$ coincide on $\mathbf{R}$.

The following theorem gives a positive answer for the above question. That is, the d-measure is a generalization of the Lebesgue measure on $\mathbf{R}$.

Lemma 1. *Let Γ be a rectifiable curve. Then $H^1(\Gamma)=D^1(\Gamma)=d^1(\Gamma)=p^1(\Gamma)=$ Length (Γ).*

Theorem 2. *$d^1(E)=\mathcal{L}^1(E)$ for $E\subset\mathbf{R}$.*

Proof. Noting that $D^1(a,b]=\mathcal{L}^1(a,b]=b-a$, we have

$$\begin{aligned} d^1(E) &= \inf\{\sum D^1(E_n) : E\subset\cup E_n, E_n \text{ are bounded}\} \\ &\leq \inf\{\sum D^1(a_n,b_n] : E\subset\cup(a_n,b_n]\} \end{aligned}$$

$$\begin{aligned} &= \inf\{\sum \ell(a_n, b_n] : E \subset \cup(a_n, b_n]\} \\ &= \mathcal{L}^1(E) \end{aligned}$$

where $\ell(\mathbf{I})$ denote the length of an interval I. Plainly $H^s(F) \leq D^s(F)$ for any bounded subset $F \subset \mathbf{R}^n$. Let $\{E_i\}$ be any bounded cover of $E \subset \mathbf{R}^n$. Then $H^s(E) \leq \sum_{i=1}^{\infty} H^s(E_i) \leq \sum_{i=1}^{\infty} D^s(E_i)$. Therefore $H^s(E) \leq d^s(E)$ for any s. Since $H^1(E) = \mathcal{L}^1(E)$, we conclude that $\mathcal{L}^1(E) = d^1(E)$.

Corollary 3. *$d^1(E) = \mathcal{L}^1(E) = H^1(E) = p^1(E)$ for $E \subset \mathbf{R}$.*

Next we compare the s-dimensional d-measure with the s-dimensional packing and Haudorff measures on $\mathbf{R}^n$.

Let $M(E, \delta)$ denote the largest number of disjoint closed balls of diameter δ with centres in E.

Lemma 4. $M(E, \delta) \geq N(E, 2\delta)$.

Proof. Let's suppose that $M(E, \delta) < \infty$. Take $B_1, B_2, \cdots, B_{M(E,\delta)}$ which are disjoint closed balls of diameter δ with centres in E. Put $B_i'(i = 1, 2, \cdots, M(E, \delta))$ be the ball of the same center with B_i and diameter 2δ. Then $\cup_{i=1}^{M(E,\delta)} B_i' \supset E$.

Theorem 5. *$H^s(E) \leq d^s(E) \leq 2^s p^s(E)$ for $E \subset R^n$.*

Proof. In the proof Theorem 2, we showed that

$$H^s(E) \leq d^s(E) \text{ for } E \subset R^n.$$

But, for bounded $E \subset \mathbf{R}^n$,

$$\begin{aligned} P^s(E) &\geq \liminf_{\delta \to 0} M(E, \delta)\delta^s \\ &\geq \liminf_{\delta \to 0} N(E, 2\delta)\delta^s \\ &= (\frac{1}{2})^s \liminf_{\delta \to 0} N(E, \delta)\delta^s \\ &= (\frac{1}{2})^s D^s(E) \end{aligned}$$

Hence

$$\begin{aligned} p^s(E) &= \inf\{\sum P^s(E_n) : \cup E_n \supset E, E_n \text{ are bounded}\} \\ &\geq (\frac{1}{2})^s \inf\{\sum D^s(E_n) : \cup E_n \supset E, E_n \text{ are bounded}\} \\ &= (\frac{1}{2})^s d^s(E). \end{aligned}$$

To investigate the relations of various dimensions we recall an old definition, a lower capacity, $\underline{\text{Cap}}(E) = \liminf_{r\to 0} \frac{\log N(E,r)}{-\log r}$ for a bounded set E in $\mathbf{R}^n$.

Proposition E [4]. *$\underline{\dim}_B(E) = \underline{Cap}(E)$ for any bounded set E in $\mathbf{R}^n$.*

Proof. Suppose that $\underline{Cap}(E) > \underline{\dim}_B(E) + \varepsilon$ for some $\varepsilon > 0$. Then there exists $\rho > 0$ such that for any $r \le \rho$, $\log N(E,r) > \log r^{-[\underline{\dim}_B(E)+\varepsilon]}$; $N(E,r) r^{\underline{\dim}_B(E)+\varepsilon} > 1$. Hence $D^{\underline{\dim}_B(E)+\varepsilon}(E) \ge 1$, which is a contradiction. Therefore $\underline{Cap}(E) \le \underline{\dim}_B(E) + \varepsilon$ for any $\varepsilon > 0$. Similarly we obtain $\underline{\text{Cap}}(E) > \underline{\dim}_B(E) - \varepsilon$ for any $\varepsilon > 0$.

Proposition F [4]. *$\widehat{\underline{\dim}}_B(E) = \underline{\dim}_{MB}(E)$ for any set E in $\mathbf{R}^n$.*

Proof. Suppose that $\widehat{\underline{\dim}}_B(E) < \underline{\dim}_{MB}(E)$. Then there exists $s \in (\widehat{\underline{\dim}}_B(E), \underline{\dim}_{MB}(E))$. So there is a sequence $\{E_n\}$ of bounded subsets of E such that $\cup_{n=1}^{\infty} E_n = E$ and $\sup_n \underline{\dim}_B(E_n) < s$. Thus $D^s(E_n) = 0$ for any n, implying $d^s(E) = 0$. It is a contradiction. Now, suppose $\underline{\dim}_{MB}(E) < \widehat{\underline{\dim}}_B(E)$. Then there exist s such that $\underline{\dim}_{MB}(E) < s < \widehat{\underline{\dim}}_B(E)$. Thus $d^s(E) = 0$. Therefore, there is a sequence $\{E_n\}$ of bounded subsets of E such that $E = \cup_{n=1}^{\infty} E_n$ and $D^s(E_n) < \infty$ for every n, hence $\underline{\dim}_B(E_n) \le s$ for every n. Thus $\widehat{\underline{\dim}}_B(E) \le s$. It is also a contradiction.

Proposition G [4]. *$\widehat{\underline{Cap}}(E) = \underline{\dim}_{MB}(E)$ for any set E in $\mathbf{R}^n$.*

Proof. It is immediate from Propositions E and F.

Corollary 6.

$$\dim_H(E) \le \underline{\dim}_{MB}(E) \le \overline{\dim}_{MB}(E) = \dim_p(E).$$

Finally we will see that for what kind of subsets of $\mathbf{R}^n$ 1-dimensional packing, Hausdorff and d-measures have same value.

Proposition 7. *A rectifiable curve Γ is a d-regular 1-set.*

Proof. Let $\ell(\Gamma)$ be the length of Γ. If Γ is rectifiable, $\ell(\Gamma) < \infty$, and since Γ has distinct end points p and q, we get $\ell(\Gamma) \ge |p-q| > 0$. Since $\ell(\Gamma) = d^1(\Gamma)$, $0 < d^1(\Gamma) < \infty$, so Γ is a d-1-set. A point x of Γ that is not an end point, divides Γ into two part $\Gamma_{p,x}$ and $\Gamma_{x,q}$. If r is sufficiently small, then removing away from x along the curve $\Gamma_{x,q}$, we reach a first point y on Γ with $|x-y| = r$. Then $\Gamma_{x,y} \subset B_r(x)$ and

$$r = |x-y| \leq \ell(\Gamma_{x,y}) = d^1(\Gamma_{x,y}) \leq d^1(\Gamma_{x,q} \cap B_r(x)).$$

Similarly, $r \leq d^1(\Gamma_{p,x} \cap B_r(x))$, so, adding $2r \leq d^1(\Gamma \cap B_r(x))$, if r is small enough. Thus

$$\underline{d}^1(\Gamma, x) = \liminf_{r \to 0} \frac{d^1(\Gamma \cap B_r(x))}{2r} \geq 1.$$

Since $\Gamma \cap B_r(x)$ is essentially the union of at most countable disjoint rectifiable curves, say $\cup_{n=1}^{\infty} \Gamma_n$,

$$d^1(\Gamma \cap B_r(x)) = \sum_{n=1}^{\infty} d^1(\Gamma_n) = \sum_{n=1}^{\infty} p^1(\Gamma_n) = p^1(\Gamma \cap B_r(x)).$$

Therefore,

$$\limsup_{r \to 0} \frac{d^1(\Gamma \cap B_r(x))}{2r} = \limsup_{r \to 0} \frac{p^1(\Gamma \cap B_r(x))}{2r}.$$

Hence $\overline{d}^1(\Gamma, x) = \overline{\Delta}^1(\Gamma, x)$. But $\overline{\Delta}^1(\Gamma, x) \leq 1$ p^1-measure almost all x in Γ (Lemmas 4.4 and 4.6[3]). Therefore $\underline{d}^1(\Gamma, x) = \overline{d}^1(\Gamma, x) = 1$ d^1-measure almost all x in Γ.

We remind that a 1-set is curve-like if it is contained in a countable union of rectifiable curves.

Proposition 8. *A curve-like set (in d-measure sense) is a d-regular 1-set.*
Proof. If F is curve-like, then $F \subset \cup_{i=1}^{\infty} \Gamma_i$, where the Γ_i are rectifiable curves. For each i and d^1-almost all $x \in F \cap \Gamma_i$, we have, using Corollary 2.6[1].

$$1 = D^1(\Gamma_i, x) = D^1(F \cap \Gamma_i, x) \leq \underline{d}^1(F \cap \Gamma_i, x) \leq \underline{d}^1(F, x)$$

and hence $1 \leq \underline{d}^1(F, x)$ for d^1-almost all $x \in F$. Clearly $\overline{\Delta}^1(F \cap \cup_{i=1}^{n} \Gamma_i, x) \leq 1$ for all $n = 1, 2, \cdots$ and for p^1-almost all $x \in F$, hence $\overline{\Delta}^1(F, x) \leq 1$ for d^1-almost all $x \in F$. Hence

$$\begin{aligned} \overline{d}^1(F, x) &= \limsup_{r \to 0} \frac{d^1(F \cap B_r(x))}{2r} \\ &= \limsup_{r \to 0} \Big(\frac{d^1(F \cap (\cup_{i=1}^{\infty} \Gamma_i) \cap B_r(x))}{2r}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{r\to 0}\big(\lim_{n\to\infty}\frac{d^1(F\cap(\cup_{i=1}^n\Gamma_i)\cap B_r(x))}{2r}\big)\\
&= \limsup_{r\to 0}\big(\lim_{n\to\infty}\frac{p^1(F\cap(\cup_{i=1}^n\Gamma_i)\cap B_r(x))}{2r}\big)\\
&= \overline{\triangle}^1(F,x)\le 1 \text{ for } d^1-\text{almost all } x\in F.
\end{aligned}
$$

Theorem 9. *$H^1(E)=d^1(E)=p^1(E)$ for p-regular set E in $\mathbf{R}^n$.*

Proof. Let E be a p-regualr set in $\mathbf{R}^n$. Then E can be the union of the subset of a countable union of rectifiable curve and a set of p^1-measure zero [3]. Say, $E\subset\cup_{i=1}^\infty\Gamma_i\cup M$. It follows from Theorem 5 that $H^1(M)=d^1(M)=p^1(M)=0$. And

$$
\begin{aligned}
p^1(E\cap\cup_{i=1}^\infty\Gamma_i) &= p^1(\lim_{n\to\infty}[E\cap\cup_{i=1}^n\Gamma_i])\\
&= \lim_n p^1(E\cap\cup_{i=1}^n\Gamma_i)\\
&= \lim_n H^1(E\cap\cup_{i=1}^n\Gamma_i)\\
&= \lim_n d^1(E\cap\cup_{i=1}^n\Gamma_i)\\
&= d^1(\lim_n[E\cap\cup_{i=1}^n\Gamma_i])\\
&= d^1(E\cap\cup_{i=1}^\infty\Gamma_i)
\end{aligned}
$$

Therefore $H^1(E)=d^1(E)=p^1(E)$.

We state a characterization of H^1 and p^1-regular sets.

Proposition H [2]. *A 1-set is regular if and only if it is the union of a curve-like set and a set of H^1-measure zero.*

Proposition I [3]. *A p-1-set is p^1-regular if and only if it is the union of a curve-like set and a set of p^1-measure zero.*

This two proposition leads us to conjecture if the similar characterization hold for d-measure.

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DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.