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NEW CRITERIA FOR MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

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Dedicated to Professor Younki Chae on his sixtieth birthday

Let $B_n(\alpha)$ be the class of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \ \ (a_{-p} \neq 0, p \in N = \{1, 2, ...\})$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$ and satisfying

$$Re\Big\{\frac{D^{n+1}f(z)}{D^n f(z)} - (p+1)\Big\} < -p\alpha \ \ (n \in N_0 = \{0, 1, 2, ...\}, |z| < 1, 0 \leq \alpha < 1),$$

where

$$D^n f(z) = \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}.$$

It is proved that $B_{n+1}(\alpha) \subset B_n(\alpha)$. Since $B_0(\alpha)$ is the class of p-valent meromorphically starlike functions of order α, all functions in $B_n(\alpha)$ are p-valent starlike. Futher property preserving integrals are considered.

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1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty}a_kz^k \quad (a_{-p}\neq 0, p\in N=\{1,2,...\}) \tag{1.1}$$

which are regular in the punctured disk $E=\{z:0<|z|<1\}$. Define

$$\begin{aligned}
(1.2)\ D^0f(z) &= f(z),\\
(1.3)\ D^1f(z) &= \frac{a_{-p}}{z^p}+(p+1)a_0+(p+2)a_1z+(p+3)a_2z^2+\cdots\\
&= \frac{(z^{p+1}f(z))'}{z^p},\\
(1.4)\ D^2f(z) &= D(D^1f(z)),
\end{aligned}$$

and for $n=1,2,...,$

$$\begin{aligned}
D^nf(z) &= D(D^{n-1}f(z)\\
&= \frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^na_{m-1}z^{m-1}\\
&= \frac{(z^{p+1}D^{n-1}f(z))'}{z^p}.
\end{aligned} \tag{1.5}$$

In this paper, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$Re\Big\{\frac{D^{n+1}f(z)}{D^nf(z)}-(p+1)\Big\}<-p\alpha \quad (z\in U=\{z:|z|<1\}) \tag{1.6}$$

for some $\alpha(0\leq\alpha<1)$ and $n\in N_0=\{0,1,2,...\}$, is meromorphically p-valent starlike in E. More precisely, it is proved that, for the class $B_n(\alpha)$ of functions in $\sum_p$ satisfying (1.6),

$$B_{n+1}(\alpha)\ \subset\ B_n(\alpha) \tag{1.7}$$

holds. Since $B_0(\alpha)$ equals $\sum_p^*(\alpha)$(the class of meromorphically p-valent starlike functions of order α [4]), the starlikeness of members of $B_n(\alpha)$ is a consequence of (1.7). Further properties preserving integrals are considered and some known results of Bajpai [1],Goel and Sohi [2] and Uralegaddi and Somanatha [6] are extended.

2. Properties of the class $B_n(\alpha)$

In proving our main results(Theorem 1 and Theorem 2 below), we shall need the following lemma due to I.S. Jack [3].

Lemma. *Let w be non-constant regular in $U = \{z : |z| < 1\}$ with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.*

Theorem 1. *$B_{n+1}(\alpha) \subset B_n(\alpha)$ for each integer $n \in N_0$.*

Proof. Let $f(z) \in B_{n+1}(\alpha)$. Then

$$Re\Big\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p+1)\Big\} < -p\alpha. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$Re\Big\{\frac{D^{n+1}f(z)}{D^{n}f(z)} - (p+1)\Big\} < -p\alpha. \tag{2.2}$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} - (p+1) = -p\frac{1+(2\alpha-1)w(z)}{1+w(z)}. \tag{2.3}$$

Clearly $w(z)$ is regular and $w(0) = 0$. The equation (2.3) may be written as

$$\frac{D^{n+1}f(z)}{D^{n}f(z)} = \frac{1+(1+2p-2\alpha p)w(z)}{1+w(z)}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity

$$z(D^n f(z))' = D^{n+1}f(z) - (p+1)D^n f(z), \tag{2.5}$$

we obtain

$$\frac{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p+1) + p\alpha}{1-\alpha} = -p\frac{1-w(z)}{1+w(z)} + \frac{2pzw'(z)}{(1+w(z))(1+(1+2p-2\alpha p)w(z))}. \tag{2.6}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z_0) = kw(z_0), \tag{2.7}$$

where $|w(z_0)| = 1$ and $k \geq 1$. From (2.6) and (2.7), we obtain

$$(2.8) \qquad \frac{\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\alpha}{1-\alpha} = -p\frac{1-w(z_0)}{1+w(z_0)} + \frac{2pkw(z_0)}{(1+w(z_0))(1+(1+2p-2\alpha p)w(z_0))}.$$

Thus

$$(2.9) \qquad Re\left\{\frac{\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\alpha}{1-\alpha}\right\} \geq \frac{1}{2(2-\alpha)} > 0,$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in B_n(\alpha)$.

Theorem 2. *Let $f(z) \in \sum_p$ satisfy the condition*

$$(2.10) \quad Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - (p+1)\right\} < -p\alpha + \frac{p(1-\alpha)}{2(p-\alpha p + c)} \quad (z \in U)$$

for a given $n \in N_0$ and $c > 0$. Then

$$(2.11) \qquad F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt$$

belongs to $B_n(\alpha)$.

Proof. Using the identities

$$(2.12) \qquad z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z)$$

and

$$(2.13) \qquad z(D^n F(z))' = D^{n+1}F(z) - (p+1)D^n F(z),$$

the condition (2.10) may be written as

$$(2.14) \quad Re\left\{\frac{\frac{D^{n+2}F(z)}{D^{n+1}F(z)} + (c-1)}{1+(c-1)\frac{D^n F(z)}{D^{n+1}F(z)}} - (p+1)\right\} < -p\alpha + \frac{p(1-\alpha)}{2(p-\alpha p + c)}.$$

We have to prove that (2.14) implies the inequality

$$(2.15) \qquad Re\left\{\frac{D^{n+1}F(z)}{D^n F(z)} - (p+1)\right\} < -p\alpha.$$

Define $w(z)$ in U by

$$\frac{D^{n+1}F(z)}{D^nF(z)}-(p+1)=-p\frac{1+(2\alpha-1)w(z)}{1+w(z)}. \tag{2.16}$$

Clearly $w(z)$ is regular and $w(0)=0$. The equation (2.16) may be written as

$$\frac{D^{n+1}F(z)}{D^nF(z)}=\frac{1+(1+2p-2\alpha p)w(z)}{1+w(z)}. \tag{2.17}$$

Differentiating (2.17) logarithmically , after simple computation we obtain

$$\frac{\frac{D^{n+2}F(z)}{D^{n+1}F(z)}+(c+1)}{1+(c-1)\frac{D^nF(z)}{D^{n+1}F(z)}}-(p+1)=-\Big[p\alpha+p(1-\alpha)\frac{1-w(z)}{1+w(z)}\Big]$$
$$+\frac{2p(1-\alpha)zw'(z)}{(c+(2p-2\alpha p+c)w(z))(1+w(z))}. \tag{2.18}$$

The remaining part of the proof is similar to that of Theorem 1.

Remarks. (1) A result of Bajpai [1, Theorem 1] turns out to be a particular case of the above Theorem 2 when $p=1, a_{-1}=1, n=0, \alpha=0$ and $c=1$. (2) For $p=1, a_{-1}=1, n=0$ and $\alpha=0$, the above Theorem 2 extends a result of Goel and Sohi [2, Corollary 1].

Theorem 3. *$f(z)\in B_n(\alpha)$ if and only if*

$$F(z)=\frac{1}{z^{1+p}}\int_0^z t^pf(t)dt \tag{2.19}$$

belongs to $B_{n+1}(\alpha)$.

Proof. From the definition of $F(z)$, we have

$$D^n(zF'(z))+(p+1)D^nF(z)=D^nf(z). \tag{2.20}$$

That is,

$$z(D^nF(z))'+(p+1)D^nF(z)=D^nf(z). \tag{2.21}$$

By using the identity(2.5), equation (2.21) reduces to $D^nf(z)=D^{n+1}F(z)$. Hence $D^{n+1}f(z)=D^{n+2}F(z)$. Therefore

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{D^{n+2}F(z)}{D^{n+1}F(z)} \tag{2.22}$$

and the result follows.

Remark. Taking $p=1$ in above theorems, we have the results of Uralegaddi and Somanatha[6].

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