NEW CRITERIA FOR MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

Sang Hun Lee and Nak Eun Cho

Dedicated to Professor Younki Chae on his sixtieth birthday

Let $B_n(\alpha)$ be the class of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in \mathbb{N} = \{1, 2, \ldots\})$$

which are regular in the punctured disk $E = \{ z : 0 < |z| < 1 \}$ and satisfying

$$Re\left\{ \frac{D^{n+1}f(z)}{D^nf(z)} - (p+1) \right\} < -p\alpha \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, |z| < 1, 0 \leq \alpha < 1),$$

where

$$D^n f(z) = \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}.$$ 

It is proved that $B_{n+1}(\alpha) \subset B_n(\alpha)$. Since $B_0(\alpha)$ is the class of $p$-valent meromorphically starlike functions of order $\alpha$, all functions in $B_n(\alpha)$ are $p$-valent starlike. Further property preserving integrals are considered.

Received March 25, 1992.

The first author supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1991 and the TGRC-KOSEF.
1. Introduction

Let $\Sigma_p$ denote the class of functions of the form

\begin{equation}
\sum_{k=0}^{\infty} a_k z^k (a_{-p} \neq 0, p \in N = \{1, 2, \ldots\})
\end{equation}

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Define

\begin{align*}
(1.2) \quad D^0 f(z) &= f(z), \\
(1.3) \quad D^1 f(z) &= \frac{a_{-p}}{z^p} + (p + 1)a_0 + (p + 2)a_1 z + (p + 3)a_2 z^2 + \cdots \\
&= \frac{(z^{p+1} f(z))'}{z^p}, \\
(1.4) \quad D^2 f(z) &= D(D^1 f(z)),
\end{align*}

and for $n = 1, 2, \ldots$,

\begin{equation}
D^n f(z) = D(D^{n-1} f(z)) \\
= \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p + m)^n a_{m-1} z^{m-1} \\
= \frac{(z^{p+1} D^{n-1} f(z))'}{z^p}.
\end{equation}

In this paper, we shall show that a function $f(z)$ in $\Sigma_p$, which satisfies one of the conditions

\begin{equation}
Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - (p + 1) \right\} < -p \alpha \quad (z \in U = \{z : |z| < 1\})
\end{equation}

for some $\alpha (0 \leq \alpha < 1)$ and $n \in N_0 = \{0, 1, 2, \ldots\}$, is meromorphically $p$-valent starlike in $E$. More precisely, it is proved that, for the class $B_n(\alpha)$ of functions in $\Sigma_p$ satisfying (1.6),

\begin{equation}
B_{n+1}(\alpha) \subset B_n(\alpha)
\end{equation}

holds. Since $B_0(\alpha)$ equals $\Sigma_p^*(\alpha)$ (the class of meromorphically $p$-valent starlike functions of order $\alpha$ [4]), the starlikeness of members of $B_n(\alpha)$ is a consequence of (1.7). Further properties preserving integrals are considered and some known results of Bajpai [1], Goel and Sohi [2] and Uralegaddi and Somanatha [6] are extended.
New criteria for meromorphically $P$-valent

2. Properties of the class $B_n(\alpha)$

In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to I.S. Jack [3].

Lemma. Let $w$ be non-constant regular in $U = \{z : |z| < 1\}$ with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at $z_0$, we have $z_0 w'(z_0) = kw(z_0)$ where $k$ is a real number, $k \geq 1$.

Theorem 1. $B_{n+1}(\alpha) \subset B_n(\alpha)$ for each integer $n \in \mathbb{N}_0$.

Proof. Let $f(z) \in B_{n+1}(\alpha)$. Then

\begin{equation}
Re\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p + 1)\right\} < -p\alpha.
\end{equation}

We have to show that (2.1) implies the inequality

\begin{equation}
Re\left\{\frac{D^{n+1}f(z)}{D^nf(z)} - (p + 1)\right\} < -p\alpha.
\end{equation}

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

\begin{equation}
\frac{D^{n+1}f(z)}{D^nf(z)} - (p + 1) = -p\frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}.
\end{equation}

Clearly $w(z)$ is regular and $w(0) = 0$. The equation (2.3) may be written as

\begin{equation}
\frac{D^{n+1}f(z)}{D^nf(z)} = \frac{1 + (1 + 2p - 2\alpha p)w(z)}{1 + w(z)}.
\end{equation}

Differentiating (2.4) logarithmically and using the identity

\begin{equation}
z(D^n f(z))' = D^{n+1} f(z) - (p + 1)D^n f(z),
\end{equation}

we obtain

\begin{equation}
\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (p + 1) + p\alpha
\end{equation}

\begin{equation}
= -p\frac{1 - w(z)}{1 + w(z)} + \frac{2pzw'(z)}{(1 + w(z))(1 + (1 + 2p - 2\alpha p)w(z))}.
\end{equation}

We claim that $|w(z)| < 1$ in $U$. For otherwise (by Jack's lemma) there exists $z_0$ in $U$ such that

\begin{equation}
z_0 w'(z_0) = kw(z_0),
\end{equation}
where $|w(z_0)| = 1$ and $k \geq 1$. From (2.6) and (2.7), we obtain

$$
\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\alpha
= \frac{1-w(z_0)}{1+\alpha} + \frac{2pkw(z_0)}{(1+w(z_0))(1+(1+2p-2\alpha p)w(z_0))}.
$$

Thus

$$
Re\left\{ \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - (p+1) + p\alpha \right\} \geq \frac{1}{2(2-\alpha)} > 0,
$$

which contradicts (2.1). Hence $|w(z)| < 1$ in $U$ and from (2.3) it follows that $f(z) \in B_n(\alpha)$.

**Theorem 2.** Let $f(z) \in \sum_\rho$ satisfy the condition

$$
Re\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} - (p+1) \right\} < -p\alpha + \frac{p(1-\alpha)}{2(p-\alpha p + c)} (z \in U)
$$

for a given $n \in N_0$ and $c > 0$. Then

$$
F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt
$$

belongs to $B_n(\alpha)$.

**Proof.** Using the identities

$$
z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z)
$$

and

$$
z(D^n F(z))' = D^{n+1} F(z) - (p+1)D^n F(z),
$$

the condition (2.10) may be written as

$$
Re\left\{ \frac{D^{n+2}F(z)}{D^{n+1}F(z)} + \frac{(c-1)}{1+(c-1)} \frac{D^n F(z)}{D^{n+1}F(z)} - (p+1) \right\} < -p\alpha + \frac{p(1-\alpha)}{2(p-\alpha p + c)}.
$$

We have to prove that (2.14) implies the inequality

$$
Re\left\{ \frac{D^{n+1}F(z)}{D^n F(z)} - (p+1) \right\} < -p\alpha.
$$
Define \( w(z) \) in \( U \) by

\[
(2.16) \quad \frac{D^{n+1}F(z)}{D^nF(z)} - (p+1) = -p\frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}.
\]

Clearly \( w(z) \) is regular and \( w(0) = 0 \). The equation (2.16) may be written as

\[
(2.17) \quad \frac{D^{n+1}F(z)}{D^nF(z)} = \frac{1 + (1 + 2p - 2\alpha p)w(z)}{1 + w(z)}.
\]

Differentiating (2.17) logarithmically, after simple computation we obtain

\[
\frac{D^{n+2}F(z)}{D^{n+1}F(z)} + \frac{(c+1)}{1 + (c-1)D^nF(z)} - (p+1) = -\left[ p\alpha + p(1 - \alpha)\frac{1 - w(z)}{1 + w(z)} \right]
+ \frac{2p(1 - \alpha)zw'(z)}{(c + (2p - 2\alpha p + c)w(z))(1 + w(z))}.
\]

(2.18)

The remaining part of the proof is similar to that of Theorem 1.

Remarks. (1) A result of Bajpai [1, Theorem 1] turns out to be a particular case of the above Theorem 2 when \( p = 1, a_{-1} = 1, n = 0, \alpha = 0 \) and \( c = 1 \).
(2) For \( p = 1, a_{-1} = 1, n = 0 \) and \( \alpha = 0 \), the above Theorem 2 extends a result of Goel and Sohi [2, Corollary 1].

**Theorem 3.** \( f(z) \in B_n(\alpha) \) if and only if

\[
(2.19) \quad F(z) = \frac{1}{z^{1+p}} \int_0^z t^p f(t) dt
\]

belongs to \( B_{n+1}(\alpha) \).

**Proof.** From the definition of \( F(z) \), we have

\[
(2.20) \quad D^n(zF'(z)) + (p + 1)D^nF(z) = D^n f(z).
\]

That is,

\[
(2.21) \quad z(D^n F(z))' + (p + 1)D^n F(z) = D^n f(z).
\]

By using the identity (2.5), equation (2.21) reduces to \( D^n f(z) = D^{n+1} F(z) \).

Hence \( D^{n+1} f(z) = D^{n+2} F(z) \). Therefore

\[
(2.22) \quad \frac{D^{n+1} f(z)}{D^n f(z)} = \frac{D^{n+2} F(z)}{D^{n+1} F(z)}
\]

and the result follows.

**Remark.** Taking \( p = 1 \) in above theorems, we have the results of Uralegaddi and Somanatha[6].
References


DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL FISHERIES UNIVERSITY OF PUSAN, PUSAN 608-737, KOREA.