AN ATOMIC DECOMPOSITION FOR TENT SPACES OVER THE UNIT BALL OF $C^n$

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Dedicated to Professor Younki Chae on his 60th birthday

1. Introduction

In 1985, R.R. Coifman and Y. Meyer and E.M. Stein introduced a family of spaces which is well adapted for the study of a variety of questions related to harmonic analysis [3]. These spaces are the “tent spaces”.

In this paper, we are going to study a duality (Theorem 1) and an atomic decomposition (Theorem 2 and Theorem 3) of tent spaces over the unit ball of $C^n$ rather than the upper half plane. In this context, Ahern-Nagel’s result was introduced in [1].

2. Preliminaries

Let $z = (z_1, z_2, \ldots, z_n)$, and $w = (w_1, w_2, \ldots, w_n)$ be two vectors in the complex $n$-space $C^n$. The inner product $<z, w>$ of $z$ and $w$ will be given by

$$<z, w> = \sum_{i=1}^{n} z_i \bar{w}_i,$$

where $\bar{w}_i$ is the complex conjugate of $w$. The corresponding norm $|z|$ of $z$ will be defined by

$$|z| = \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2}.$$
The open unit ball in $C^n$ will be denoted by $B$, i.e.,

$$B = \{z \in C^n : |z| < 1\}.$$

The boundary of $B$ is the sphere $S$, that is, the set

$$S = \{z \in C^n : |z| = 1\}.$$

We let $\sigma$ denote the area measure on $S$ throughout paper. The Lebesgue measure $v$ and the measure $\sigma$ are related by the following formula

$$\int_{C^n} f \, dv = C_n \int_0^\infty r^{2n-1} \, dr \int_S f(r \zeta) \, d\sigma(\zeta),$$

for a constant $C_n > 0$. For $\zeta, \eta \in S$, and $\delta > 0$, let

$$\rho(\zeta, \eta) = |1 - \langle \zeta, \eta \rangle|,$$

and

$$\beta(\zeta, \delta) = \{\eta \in S : \rho(\eta, \zeta) = |1 - \langle \eta, \zeta \rangle| < \delta\}.$$

Then it is easy to check that $\rho$ defines a pseudo-metric on $S$ and that the triple $(S, \rho, d\sigma)$ becomes a space of homogeneous type [5]. The ball $\beta(\zeta, \delta)$ has basically a different structure when it is compared with Euclidean balls. This new ball $\beta(\zeta, \delta)$ is nonisotropic. Note that $\sigma(\beta(\zeta, \delta))$ is roughly proportional to $\delta^n$. (See Rudin [9])

Now consider an approach region associated with this ball.

For $\alpha > 1$, and $\zeta \in S$, let

$$A_{\alpha}(\zeta) = \{z \in B : |1 - \langle z, \zeta \rangle| < \alpha(1 - |z|)\}.$$

Then $A_{\alpha}(\zeta)$ is called an admissible approach region.

Let $f \in L^1(d\sigma)$. For $\xi \in S$, we define

$$\mathcal{M}(f)(\xi) = \sup_{\delta > 0} \frac{1}{\sigma(\beta(\xi, \delta))} \int_{\beta(\xi, \delta)} |f(\zeta)| \, d\sigma(\zeta).$$

The function $\mathcal{M}$ is called the Hardy - Littlewood's maximal function.

For a closed subset $F \subset S$, and $\alpha > 1$, let

$$R^{(\alpha)}(F) = \bigcup \{A_{\alpha}(\zeta) : \zeta \in F\}.$$

The tent $T(O)$ over an open subset $O = F^c$ is defined by the complement of $R^{(4)}(F)$. Precisely this is a complex version of tents.
Let $f$ be a function defined on the unit ball $B$. Define a functional $A_q(f)$, for $\zeta \in S$, by

$$A_q(f)(\zeta) = \left[ \int_{A_4(\zeta)} |f(z)|^q \frac{1}{(1-|z|)^{1+\frac{1}{n}}} dv(z) \right]^{1/q}$$

if $q < \infty$.

and

$$A_{\infty}(f)(\zeta) = \sup_{z \in A_4(\zeta)} |f(z)|$$

if $q = \infty$ \hspace{1cm} (2.1)

The "tent space" $T^p_q(B)$ is defined as the space of function $f$, so that $A_q(f) \in L^p(d\sigma)$, when both $p$ and $q$ are finite. The resulting equivalences classes are then equipped with a norm, $\|f\|_{T^p_q} = \|A_q(f)\|_p$.

The case $q = \infty$ (with $p = 1$), requires a natural modification since (2.1) is a "sup norm".

The tent space $T^1_{\infty}(B)$ will denote the class of all $f$ which are continuous in $B$, for which $A_{\infty}(f)(\zeta) \in L^1(d\sigma)$ and for which $\|f - f\|_{T^1_{\infty}} \to 0$, where $\epsilon \to 0$, $\epsilon < 1$, with $f_\epsilon(z) = f(\epsilon z)$.

For $0 < p \leq 1$, $1 < q < \infty$, a function $a(z)$ defined on $B$ is said to be a $(p,q)$-atom if

(i) $a(z)$ is supported on the tent $T(\beta(\zeta, \delta))$ of a ball $\beta(\zeta, \delta)$

(ii) $\int_{T(\beta(\zeta, \delta))} |a(z)|^q \frac{dv(z)}{1-|z|^q} \leq \sigma(\beta(\zeta, \delta))^{1-q/p}$,

and a function $a(z)$ defined on $B$ is said to be a $(1,\infty)$-atom if

(i) $a(z)$ is supported on the tent $T(\beta(\zeta, \delta))$ of a ball $\beta(\zeta, \delta)$

(ii) $\sup_{z \in T(\beta(\zeta, \delta))} |a(z)| \leq \frac{1}{\sigma(\beta)}$.

3. Duality and Atomic decomposition for tent spaces

Suppose $d\mu$ is a measure defined on $\bar{B}$. Then it is said to satisfy Carleson's condition if

$$\sup_{\beta(\zeta, \delta)} \frac{1}{\sigma(\beta(\zeta, \delta))} \int_{T(\beta(\zeta, \delta))} |d\mu(z)| \leq C < \infty.$$ 

Lemma 1. ([4], Whitney decomposition) Let $O \subset S$ be an open set. Then there are a positive constant $M$, $C_1 > 1$, $C_2 > 1$ and $C_3 < 1$, which depend only on the dimension $n$, and a sequence $\{\beta(\zeta_i, \delta_i)\}$ of balls such that

(a) $\cup \beta(\zeta_i, \delta_i) = O$,

(b) $\beta(\zeta_i, C_2 \delta_i) \subset O$ and $\beta(\zeta_i, C_1 \delta_i) \cap O^c \neq \phi$,

(c) the balls $\beta(\zeta_i, C_3 \delta_i)$ are pairwise disjoint,
(d) no point in \( O \) lies in more than \( M \) of the balls \( \beta(\zeta_i, C_2 \delta_i) \).

Although the constant \( C \) is used unambiguously in several places, it does not depend on particular functions.

**Lemma 2.** Suppose that \( f(z) \) is a function on \( B \) and that \( A_\infty(f) \) is a lower semicontinuous function. Let \( \mu \) be a Carleson measure. Then, for each \( \lambda > 0 \), there exists a constant \( C \) so that

\[
|\mu|\{z \in B : |f(z)| > \lambda\} \leq C \sigma\{\zeta \in S : A_\infty(f)(\zeta) > \lambda\}
\]

(3.1)

for all \( f \) and \( \lambda > 0 \).

**Proof.** Since \( A_\infty(f) \) is lower semicontinuous, the set \( \{\zeta \in S : A_\infty(f)(\zeta) > \lambda\} \) is open in \( S \). Let \( \{\zeta \in S : A_\infty(f)(\zeta) > \lambda\} = \bigcup_i \beta(\zeta_i, \delta_i) \) be a Whitney decomposition. Then we have \( \{z \in B : |f(z)| > \lambda\} \subset \bigcup_i T(\beta(\zeta_i, C \delta_i)) \), where \( C \) is sufficiently large. Since \( \mu \) is a Carleson measure, it follows from doubling property and Whitney decomposition that

\[
|\mu|\{z \in B : |f(z)| > \lambda\} \leq |\mu|\bigcup_i T(\beta(\zeta_i, C \delta_i)) \\
\leq C \sum_i \sigma(\beta(\zeta_i, C \delta_i)) \\
\leq C \sum_i \sigma(\beta(\zeta_i, C' \delta_i)), \text{ where } C' < 1 \\
\leq C \sigma(\bigcup_i \beta(\zeta_i, \delta_i)) \\
= C \sigma\{\zeta \in S : A_\infty(f)(\zeta) > \lambda\}.
\]

**Theorem 1.** The dual space of \( T_\infty^1(B) \) is the space of Carleson measures; more precisely, the pairing

\[
(f, d\mu) \rightarrow \int_B |f(z)| d\mu(z)
\]

with \( f \) ranging over functions which are \( T_\infty^1(B) \) and are continuous in \( \bar{B} \) and \( d\mu \) over Carleson measures, realizes the duality of \( T_\infty^1(B) \) with Carleson measures.

**Proof.** If \( d\mu \) is a Carleson measure on \( B \) and if \( f \) is continuous in \( \bar{B} \), then an integration of (3.1) gives

\[
\int_B |f(z)||d\mu(z)| \leq C \left( \sup_\beta \frac{1}{\sigma(\beta)} \int_{T(\beta)} |d\mu| \right) \cdot ||A_\infty(f)||_{L^1(d\sigma)}
\]

(3.2)
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and the fact that every Carleson measure gives a bounded linear functional on $T^1_\infty(B)$ follows from the inequality (3.2). Conversely, a bounded linear functional on $T^1_\infty(B)$ gives a family of bounded linear functionals on $C(K)$ which is the space of continuous functions with compact support in $\bar{B}$. This induces a measure $d\mu$ on $\bar{B}$. To show $d\mu$ is a Carleson measure, write $d\mu = \eta |d\mu|$ and put $f = \tilde{\eta} \chi_{T(\beta)}$, with $\beta \subset S$. Let $\{f_n\}$ be a sequence of continuous functions with compact support which converges to $f = \tilde{\eta} \chi_{T(\beta)}$, in the sense of $T^1_\infty(B)$ norm-convergence. Since $A_\infty(f) = \chi_\beta$, by the continuity of linear functional, we get

$$\int_{T(\beta)} d|\mu| \leq |\int_{T(\beta)} \tilde{\eta} d\mu|$$

$$= |\int_{T(\beta)} f d\mu|$$

$$\leq C \|A_\infty(f)\|_{L^1(d\sigma)}$$

$$= C \sigma(\beta).$$

However the $C(K)$ is dense in $T^1_\infty(B)$. The proof is complete.

Next, we need some notations before proving theorem 2 and theorem 3. Let $F$ be a closed subset of $S$. Let $\gamma$ be fixed and $0 < \gamma < 1$. Then we say that a point $\zeta \in S$ has a global $\gamma$-density with respect to $F$ if

$$\frac{\sigma[F \cap \beta(\zeta, \delta)]}{\sigma[\beta(\zeta, \delta)]} \geq \gamma$$

for all $\delta > 0$. Let $\gamma(F)$ be the set of all the points of a global $\gamma$-density with respect to $F$. Note that $\gamma(F)$ is a closed set and

$$\gamma(F)^c = \{\zeta \in S : \mathcal{M}(\chi_{F^c})(\zeta) > 1 - \gamma\},$$

where $\chi_{F^c}$ is the characteristic function of the open set $F^c$.

**Lemma 3**[10]. Let $F$ be a closed subset of $S$. Then there is a constant $C_\gamma$ so that

$$\sigma[\gamma(F)^c] \leq C_\gamma \sigma(F^c).$$

**Lemma 4.** Suppose $O$ is an open subset of $S$. If $z \in T(O)$ then there is a constant $\alpha > 0$ such that $\beta(z/|z|, \alpha(1 - |z|)) \subset O$. The converse holds for $\alpha > 10$. 
Proof. If \( z \in T(O) \), then by the definition of \( T(O) \), \( z \notin A_4(\zeta) \) for all \( \zeta \in F = O^c \). That is, \( |1 - < z, \zeta > | \geq 4(1 - |z|) \) for all \( \zeta \in F \). On the other hand, if \( \zeta \in \beta(z/|z|, 1 - |z|) \), then \( |1 - < z, \zeta > | \leq 2[|1 - z/|z|, \zeta > | + |1 - < z, z/|z| > |] < 4(1 - |z|) \). Therefore \( \zeta \in O \).

Conversely, \( z \in T(\beta(z/|z|, \alpha(1 - |z|))) \) for all \( \alpha > 10 \). We have \( z \in T(O) \).

Theorem 2. Let \( f \in T_{\infty}^1(B) \). Then there exists a constant \( C \), a sequence \( \{a_j\} \) of \((1, \infty)\)-atoms and a sequence \( \{\lambda_j\} \) of positive numbers so that

\[
|f(z)| \leq \sum_{j=1}^{\infty} \lambda_j |a_j(z)|,
\]

and

\[
\sum_{j=1}^{\infty} \lambda_j \leq C\|A_\infty(f)\|_{L^1(d\sigma)}.
\]

Proof. Define, for each integer \( k \),

\[
O_k = F_k^c = \{ \zeta \in S: A_\infty(f)(\zeta) > 2^k \}.
\]

Observe that

\[
O_{k+1} \subset O_k,
\]

\[
T(O_{k+1}) \subset T(O_k),
\]

and \( \cup_k T(O_k) \) contains the support of \( f \). Let \( O_k = \cup_{j=1}^{\infty} \beta_{k,j}(\zeta_{k,j}, \delta_{k,j}) = \cup_{j=1}^{\infty} \beta_{k,j} \) be a Whitney decomposition of the open set \( O_k \). Let \( \tilde{\beta}_{k,j} = \beta(\zeta_{k,j}, CM\delta_{k,j}) \), where \( M \) is given in (d) of Lemma 1 and \( C \) will be choosen sufficiently large in a moment. Thus we have

\[
T(O_k) - T(O_{k+1}) = \cup_j \Delta_j^k,
\]

where

\[
\Delta_j^k = T(\tilde{\beta}_{k,j}) \cap (T(O_k) \cap T(O_{k+1})^c).
\]

We distinguish cases (a) and (b).

Case (a). For every \( k, O_k \neq S \).

If we let \( \chi_{k,j} \) be the characteristic function of the set \( \Delta_j^k \), then we have

\[
|f(z)| \leq \sum_{k,j} |f(z)| \chi_{k,j}(z) = \sum_{k,j} |a_{k,j}| \lambda_{k,j}.
\]
where
\[
\begin{align*}
a_{k,j} &= f \cdot \chi_{k,j} \sigma(\tilde{\beta}_{k,j})^{-1} 2^{-k-1}, \\
\lambda_{k,j} &= \sigma(\tilde{\beta}_{k,j}) 2^{k+1}.
\end{align*}
\]

Now \(a_{k,j}\) is a \((1, \infty)\)-atom associated with the ball \(\tilde{\beta}_{k,j}\) and we have
\[
\sum_{k,j} \lambda_{k,j} = \sum_{k,j} \sigma(\tilde{\beta}_{k,j}) 2^{k+1} \\
= \sum_k 2^{k+1} \sum_j \sigma(\tilde{\beta}_{k,j}) \\
\leq C \sum_k 2^{k+1} \sum_j \sigma(\beta) \\
\leq C \sum_k 2^{k+1} \sigma(O_k) \\
\leq C \|A_{\infty}(f)\|_{L^1(d\sigma)}.
\]

Case (b). There is an integer \(k\) such that \(O_k = S\). Without loss of generality, we may assume \(k = 1\). Then \(O_1 = S\) and \(O_k \neq S\) if \(k > 1\). Let
\[
\Delta_1 = B \cap T(O_2)^c, \\
a_1 = \frac{1}{4} \cdot f \cdot \chi_{\Delta_1} \sigma(S)^{-1}, \\
\lambda_1 = 4 \sigma(S).
\]
Then \(a_1\) is a \((1, \infty)\)-atom supported on \(B\). For \(k > 1\), define \(\chi_{k,j}, \lambda_{k,j}\) and \(a_{k,j}\) as before. Then we have
\[
|f(z)| \leq |f(z)|\chi_{\Delta_1}(z) + \sum_{k,j} |f(z)|\chi_{k,j}(z) \\
= \lambda_1 |a_1| + \sum_{k,j} \lambda_{k,j} |a_{k,j}|.
\]
Finally
\[
\lambda_1 = 4 \sigma(S) \\
\leq C \|A_{\infty}(f)\|_{L^1(d\sigma)},
\]
and for \(k > 1\), we have as before
\[
\sum \lambda_{k,j} \leq C \|A_{\infty}(f)\|_{L^1(d\sigma)}.
\]
Lemma 5[10]. Suppose \( \alpha > 1 \) be given. There exists a constant \( C_{\alpha, \gamma} \) and \( \gamma, 0 < \gamma < 1 \), sufficiently close to 1, so that whenever \( F \) is a closed subset of \( S \) and \( \Phi \) is a nonnegative function defined on the unit ball \( B \), then

\[
\int_{F(\alpha) \gamma(F)} \Phi(z)(1 - |z|)^{\alpha} dv(z) \leq C_{\alpha, \gamma} \int_F \int_{A_4(\zeta)} \Phi(z) dv(z) d\sigma(\zeta).
\]

Theorem 3. Let \( f \in T^p_q(B), \ 0 < p \leq 1, \ 1 < q < \infty \). Then there exists a constant \( C \), a sequence \( \{a_j\} \) of \( (p, q) \)-atoms, and a sequence \( \{\lambda_j\} \) of positive numbers so that

\[
|f(z)| \leq \sum_{j=1}^{\infty} \lambda_j |a_j(z)|,
\]

and

\[
\sum_{j=1}^{\infty} \lambda_j^p \leq C \|A_q(f)\|_{L^p(d\sigma)}^p.
\]

Proof. Define, for each integer \( k \),

\[
O_k = F_k^c = \{ \zeta \in S : A_q(f)(\zeta) > 2^k \}.
\]

Let \( O_k^* = \gamma(F_k)^c \). Then by the property of a global \( \gamma \)-density (with \( \gamma \) sufficiently close to 1), it follows that

\[
O_k^* = \{ \zeta \in S : M(\chi_{O_k})(\zeta) > 1 - \gamma \}.
\]

By Lemma 3, we get that \( \sigma[O_k^*] \leq C_\gamma \sigma[O_k] \). Observe that for each integer \( k \),

\[
O_{k+1} \subset O_k, \quad O_k \subset O_k^*, \quad T(O_k) \subset T(O_k^*),
\]

and \( \cup T(O_k^*) \) contains the support of \( f \). Since \( \gamma(F_k) \) is a closed subset of \( S \), \( O_k^* \) is an open set. Let \( O_k^* = \bigcup_{j=1}^{\infty} \beta(\zeta_{k,j}, \delta_{k,j}) = \bigcup_{j=1}^{\infty} \beta_{k,j} \) be a Whitney decomposition. Let \( \tilde{\beta}_{k,j} = \beta(\zeta_{k,j}, CM\delta_{k,j}) \), where \( M \) is given in (d) of Lemma 1 and \( C \) will be chosen sufficiently large in a moment. By Lemma
4, we know that \( z \in T(\Omega_k^*) \) implies that \( \beta(z/|z|, 1 - |z|) \subseteq \Omega_k^* \). Let \( z/|z| \in \beta_{k,j^*} \), for some \( j^* \). If \( \eta \in \beta(\zeta_{k,j^*}, M\delta_{k,j^*}) \cap \gamma(F_k) \), then

\[
1 - |z| \leq |1 - z/|z||, \eta > | \leq 2[|1 - z/|z||, \zeta_{k,j^*} > | + |1 - z/|z||, \zeta > |] \leq 2(1 + M)\delta_{k,j^*}.
\]

Hence if \( \zeta \in \beta(z/|z|, \alpha(1 - |z|)) \) for \( \alpha > 0 \), then it follows by (3.3) that

\[
|1 - \zeta_{k,j^*}, \zeta > | \leq 2[|1 - \zeta_{k,j^*}, z/|z| > | + |1 - z/|z||, \zeta > |] < 2[\delta_{k,j^*} + \alpha(1 - |z|)] < 2[\delta_{k,j^*} + 2\alpha(1 + M)\delta_{k,j^*}] = 2(1 + 2\alpha + 2\alpha M)\delta_{k,j^*}.
\]

If we choose \( C \) so that \( 2(1 + 2\alpha + 2\alpha M) < CM \), then it follows that

\[
\beta(z/|z|, \alpha(1 - |z|)) \subseteq \beta(\zeta_{k,j^*}, CM\delta_{k,j^*}) \equiv \tilde{\beta}_{k,j^*},
\]

and so for \( \alpha > 10 \),

\[
z \in T(\beta(z/|z|, \alpha(1 - |z|))) \subseteq T(\tilde{\beta}_{k,j^*}).
\]

Thus we can write

\[
T(\Omega_k^*) \cap T(\Omega_{k+1}^*)^c = \bigcup_j \Delta_{k,j},
\]

where

\[
\Delta_{k,j} = T(\tilde{\beta}_{k,j}) \cap [T(\Omega_k^*) \cap T(\Omega_{k+1}^*)^c].
\]

We distinguish two cases (a) and (b).

Case (a). For every \( k \), \( O_k^* \neq S \). If we let \( \chi_{k,j} \) be the characteristic function of the set \( \Delta_{k,j} \), then we have

\[
|f(z)| \leq \sum_{k,j} |f(z)|\chi_{k,j}(z) \equiv \sum_{k,j} |a_{k,j}|\lambda_{k,j},
\]

where

\[
a_{k,j}(z) = f(z) \cdot \chi_{k,j}(z)\sigma(\tilde{\beta}_{k,j})^{1/q-1/p} \left[ \int_{\Delta_{k,j}} |f(z)|^q \frac{dv(z)}{1 - |z|} \right]^{-1/q}.
\]
and
\[ \lambda_{k,j} = \sigma(\tilde{\beta}_{k,j})^{-1/q+1/p} \left[ \int_{\Delta_{k,j}} |f(z)|^q \frac{dv(z)}{1-|z|} \right]^{1/q}. \]

Now \( a_{k,j} \) is a \((p,q)\)-atom associated with the ball \( \tilde{\beta}_{k,j} \). Put \( F = O_{k+1}^c \),
\( R^*(\gamma(F)) = T(O_{k+1}^*)^c, \gamma(F) = (O_{k+1}^*)^c \), and \( \Phi(z) = |f(z)|^q \frac{1}{(1-|z|)^{1+n}} \chi_{T(\tilde{\beta}_{k,j})}(z) \)
and then apply Lemma 5 to get that
\[ \int_{\Delta_{k,j}} |f(z)|^q \frac{dv(z)}{1-|z|} \leq \int_{T(\tilde{\beta}_{k,j}) \cap T(O_{k+1}^*)^c} |f(z)|^q \frac{dv(z)}{1-|z|} \]
\[ = \int_{T(O_{k+1}^*)^c} \chi_{T(\tilde{\beta}_{k,j})}(z) |f(z)|^q \frac{dv(z)}{1-|z|} \]
\[ \leq C_\gamma \int_{O_{k+1}^*} \int_{A_k(\zeta)} |f(z)|^q \chi_{T(\tilde{\beta}_{k,j})}(z) \frac{dv(z)d\sigma(\zeta)}{(1-|z|)^{1+n}} \]
\[ \leq C_\gamma \int_{O_{k+1}^*} \int_{A_k(\zeta)} A_q(f)^q(\zeta) d\sigma(\zeta) \]
\[ \leq C_\gamma 2^{q(k+1)} \sigma(\tilde{\beta}_{k,j}). \]

Since
\[ \sigma(\tilde{\beta}_{k,j}) \leq C \sigma(\beta_{k,j}) \]
by the doubling property of \( \beta_{k,j} \), we have
\[ \sum_{k,j} \lambda_{k,j}^p = \sum_{k,j} \sigma(\tilde{\beta}_{k,j})^{1-p/q} \left[ \int_{\Delta_{k,j}} |f(z)|^q \frac{dv(z)}{1-|z|} \right]^{p/q} \]
\[ \leq C \sum_{k,j} 2^{p(1-p/q)} \sigma(\beta_{k,j})^{1-p/q} \sigma(\beta_{k,j})^{p/q} \]
\[ \leq C \sum_{k,j} 2^{p(1-p/q)} \sigma(\beta_{k,j}) \]
\[ \leq C \sum_k 2^{p(1-p/q)} \sigma(O_k^*) ( \text{ by Lemma 1 }) \]
\[ \leq C \sum_k 2^{p(1-p/q)} \sigma(O_k) ( \text{ by Lemma 3 }) \]
\[ \leq C \| A_q(f) \|_{L^p(d\sigma)}^p. \]

Case (b). There is an integer \( n \) such that \( O_n^* = S \). Without loss of
generality, we may assume \( n = 1 \). Then \( O_1^* = S \), and \( O_k^* \neq S \) if \( k > 1 \).
Let
\[ \Delta_1 = B \cap T(O_2^*)^c, \]
\[ \lambda_1 = \sigma(S)^{-1/q+1/p} \left[ \int_{\Delta_1} |f(z)|^q \frac{dv(z)}{1 - |z|} \right]^{1/q}, \]
and
\[ a_1(z) = f(z) \cdot \chi_{\Delta_1}(z) \cdot \sigma(S)^{-1/p+1/q} \left[ \int_{\Delta_1} |f(z)|^q \frac{dv(z)}{1 - |z|} \right]^{-1/q}, \]
where \( \chi_{\Delta_1} \) is the characteristic function of \( \Delta_1 \). Then \( a_1 \) is a \((p,q)\)-atom supported on \( B \). For \( k > 1 \), define \( a_{k,j} \) as before. Then we have
\[ |f(z)| \leq |f(z)|\chi_{\Delta_1}(z) + \sum_{k \geq 2, j} |f(z)|\chi_{k,j}(z) \]
\[ = \lambda_1 |a_1| + \sum_{k \geq 2, j} \lambda_{k,j} |a_{k,j}|. \]
and apply Lemma 5 to obtain
\[ \lambda_1^p = \sigma(S)^{-p/q+1} \left[ \int_{\Delta_1} |f(z)|^q \frac{dv(z)}{1 - |z|} \right]^{p/q} \]
\[ \leq C \sigma(S)^{-p/q+1} \left[ \int_{O_2^*} \int_{A_q(\zeta)} |f(z)|^q \frac{dv(z)}{(1 - |z|)^{1+n} d\sigma(\zeta)} \right]^{p/q} \]
\[ \leq C \sigma(S) \]
\[ \leq C \sigma(O_1) \text{ (by Lemma 3)} \]
\[ \leq C \|A_q(f)\|_{L^p(d\sigma)}^p. \]
For \( k \geq 2 \), we have as before
\[ \sum_{k,j} \lambda_{k,j}^p \leq C \|A_q(f)\|_{L^p(d\sigma)}. \]

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