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# WREATH PRODUCT OF REGULAR $*$-SEMIGROUPS

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Dedicated to Professor Younki Chae on his 60th birthday

## 1. Introduction

The concept of algebraic regular $*$ - semigroup was introduced by McAlister ([2]). In recent years, some authers established many characterizations of such object [2], [9], [10], [11], and [12]. In this paper, we first discuss a topological regular $*$ - semigroup of continuous functions from a locally compact space into a topological regular $*$ - semigroup. And we establish the wreath product of topological regular $*$ - semigroups as one of the semidirect products of topological semigroups. Many properties concerned with the wreath product of algebraic semigroups are well known in [8], [13] and related papers.

## 2. Preliminaries

Throughout, all topological spaces will assume Hausdorff spaces. A *semigroup* is a nonempty set $S$ together with an associative multiplication. An element $e$ of a semigroup $S$ is called an *idempotent* if $e^2 = e$.

A *topological semigroup* is a Hausdorff space $S$ together with a continuous associative multiplication.

**Definition**([10]). A semigroup $S$ with a unary operation $* : S \to S$ is called a $* -$ *semigroup* if it satisfies

(1) $(x^*)^* = x$ for all $x \in S$,

(2) $(xy)^* = y^*x^*$ for all $x, y \in S$.

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A $*$ $-$ *semigroup* $S$ is called a regular $*$ - semigroup if $x = xx^*x$ for all $x \in S$.

Let $S$ be a $*$ - semigroup. An idempotent $e \in S$ is called a *projection* if $e^* = e$. We denote the set of all projections of $S$ by $P(S)$.

Note that if $S$ is a regular $*$ - semigroup then $xx^*$ and $x^*x$ are projections of $S$ for each $x \in S$.

**Definition**. *A topological regular* $*$ *- semigroup* is a topological semigroup $S$ which is a regular $*$ - semigroup and the unary operation $*$ on $S$ is a continuous function.

## 3. Regular * - semigroup of Continuous Functions

If $X$ and $Y$ are Hausdorff spaces, then $C(X,Y)$ denotes the set of all continuous functions from $X$ into $Y$. For Hausdorff spaces $X$ and $Y$, we will be assumed the remainder that $C(X,Y)$ is assigned the compact - open topology.

Let $S$ and $T$ be topological semigroups. The *pointwise multiplication* on $C(S,T)$ is defined by $(fg)(x) = f(x)g(x)$ for all $x \in S$.

**Theorem 3.1**([6]). *Let $S$ be a locally compact space and let $T$ be a topological semigroup. Then $C(S,T)$ with the pointwise multiplication is a topological semigroup.*

**Theorem 3.2**. *Let $S$ be a locally compact space and let $T$ be a topological regular $*$ - semigroup. Then $C(S,T)$ with the pointwise multiplication is a topological regular $*$ - semigroup.*

*Proof.* In view of Theorem 3.1., $C(S,T)$ is a topological semigroup. We establish that $C(S,T)$ is a regular $*$ - semigroup and the unary operation on $C(S,T)$ is continuous; Let $\phi : T \to T$ be the unary operation. For each $f \in C(S,T)$, let $f^* = \phi \circ f$, that is $f^*(x) = (\phi \circ f)(x) = \phi(f(x)) = f(x)^*$ for all $x \in S$. Then $f^* \in C(S,T)$. For $x \in S$, $(f^*)^*(x) = (\phi \circ (\phi \circ f))(x) = (f(x)^*)^* = f(x)$. So $(f^*)^* = f$. Let $g \in C(S,T)$. Then $(fg)^*(x) = (\phi \circ (fg))(x) = ((fg)(x))^* = (f(x)g(x))^* = g(x)^*f(x)^* = (\phi \circ g)(\phi \circ f)(x) = (g^*f^*)(x)$ for all $x \in S$. So, $(fg)^* = g^*f^*$. Thus $C(S,T)$ is a $*$ - semigroup. Moreover, for $x \in S$, $(ff^*f)(x) = f(x)f^*(x)f(x) = f(x)f(x)^*f(x) = f(x)$. So $ff^*f = f$ for all $f \in C(S,T)$. Hence $C(S,T)$ is a regular $*$ - semigroup. To prove that the unary operation on $C(S,T)$ is continuous, let $\rho : C(S,T) \to C(S,T)$ be the unary operation. Then $\rho(f) = f^* = \phi \circ f$. Let $K$ be a compact subset of $S$, $W$

an open subset of $T$, $f \in C(S,T)$, and $f^* = \rho(f)(K) \in N(K,W)$. Then $(\phi \circ f)(K) = f^*(K) = \rho(f)(K) \subset W$. Hence $f(K) \subset \phi^{-1}(W)$, and hence $f \in N(K, \phi^{-1}(W))$, where $\phi^{-1}(W)$ is open subset of $T$ because the unary operation $\phi$ is continuous. If $g \in N(K, \phi^{-1}(W))$, then $g(K) \subset \phi^{-1}(W)$. So $\rho(g)(K) = g^*(K) = (\phi \circ g)(K) \subset W$, and so $\rho(g) \in N(K,W)$. Thus $\rho(N(K, \phi^{-1}(W))) \subset N(K,W)$. Hence $\rho$ is continuous. Therefore $C(S,T)$ is a topological regular * - semigroup.

## 4.Wreath Product of Regular * - semigroups

If S is a [topological] semigroup, then we use End $(S)$ to denote *the set of [continuous] endomorphisms* of $S$. Note that if $S$ is a [locally compact] semigroup then End $(S)$ [with the relative topology of $C(S,T)$ ] is a [topological] semigroup under the composition of [continuous] homomorphisms ([4]).

**Definition**. Let $S$ be a [locally compact] semigroup, $T$ a [topological] semigroup. If there exist a [continuous] homomorphism $\phi : T \to End(S)$, then we define the *semidirect product* $S \times_\phi T$ of $S$ and $T$ to be $S \times T$ [with the product topology] together with multiplication $((s_1,t_1),(s_2,t_2)) \to (s_1\phi(t_1)(s_2), t_1t_2)$.

**Lemma 4.1**. *Let $S$ be a [locally compact] simigroup, $T$ a [topological] semigroup, and $\phi : T \to End(S)$ a [continuous] homomorphism. Then $S \times_\phi T$ is a [topological] semigroup.* (See [4]).

**Definition**. Let $S$ and $T$ be semigroups and let $S^T$ be the set of all functions from $T$ into $S$. *The wreath product* $S \odot T$ of $S$ and $T$ is the set $S^T \times T$ with multiplication defined by $((f,a),(g,b)) \to (fg_a, ab)$ for all $f,g \in S^T$ and $a,b \in T$, where $(fg_a)(x) = f(x)g(xa)$ for all $x \in T$.

*Remark.* Suppose $S$ and $T$ are semigroups. Then the set $S^T$ of all functions from $T$ into $S$ is a semigroup under the pointwise multiplication. Define $\phi : T \to End(S^T)$ by $\phi(t) = \phi \circ \rho_t$, where $\rho_t$ is a right translation by $t$ in $T$. Then $\phi$ is a homomorphism. Hence the wreath product $S \odot T$ of $S$ and $T$ is $S^T \times_\phi T$, and hence $S \odot T = S^T \times T$ with multiplication given by $((f,a),(g,b)) \to (fg \circ \rho_a, ab)$.

In view of Lemma 4.1 and Remark, the following theorems are easily obtained.

**Theorem 4.2**([6]). *Let $S$ and $T$ be semigroups. Then the wreath product*

*$S \odot T$ of $S$ and $T$ is a semigroup.*

**Theorem 4.3**([6]). *Let $S$ be a topological semigroup and let $T$ be a locally compact topological semigroup. Suppose that the semigroup $S^T$ of all continuous functions from $T$ into $S$ is locally compact and suppose $\phi : T \to End(S^T)$ given by $\phi(a)(f) = f \circ \rho_a$ is continuous. Then the wreath product $S \odot T$ of $S$ and $T$ is a topological semigroup.*

**Theorem 4.4**. *Let $S$ and $T$ be regular $*$ - semigroups. If $f(xe) = f(x)$ for all $e \in P(T), x \in T$ and $f \in S^T$, then the wteath product $S \odot T$ of $S$ and $T$ is a regular $*$ - semigroup.*

*Proof.* In view of Lemma 4.1., $S \odot T$ is a semigroup. Let $(f,a) \in S \odot T = S^T \times T$. Define $(f,a)^* = (g, a^*)$ such that $g(x) = (f(xa^*))^*$ for all $x \in T$. Then $(f,a)^* = (g,a^*)^* = (h,(a^*)^*)$ such that $g(x) = f(xa^*)^*$ and $h(x) = g(x(a^*)^*)^*$ for all $x \in T$. So, $h(x) = g(xa)^* = (f(xaa^*)^*)^* = (f(x)^*)^* = f(x)$ for all $x \in T$,and so $h = f$. Hence $((f,a)^*)^* = (f,a)$. And let $(g,b) \in S \odot T = S^T \times T$. Then $((f,a)(g,b))^* = (fg_a, ab)^* = (h,(ab)^*) = (h, b^*a^*)$ such that $h(x) = fg_a(x(ab)^*)^*$ for all $x \in T$. So $h(x) = (f(x(ab)^*)g(x(ab)^*a))^* = (f(xb^*a^*)g(xb^*a^*a))^* = g(xb^*)^*f(xb^*a^*)^*$. On the other hand, $(g,b)^*(f,a)^* = (k,b^*)(l,a^*) = (kl_{b^*}, b^*a^*)$ such that $k(x) = g(xb^*)^*$ and $l(x) = f(xa^*)^*$ for all $x \in T$. So, $(kl_{b^*})(x) = k(x)l(xb^*) = g(xb^*)^*f(xb^*a^*)^*$ for all $x \in T$. Hence $h = kl_{b^*}$, and hence $((f,a)(g,b))^* = (g,b)^*(f,a)^*$ for all $(f,a),(g,b) \in S \odot T$. Thus $S \odot T$ is a $*$ - semigroup. Next, *let*$(f,a) \in S \odot T = S^T \times T$ and let $(f,a)^* = (g,a^*)$ such that $g(x) = f(xa^*)^*$ for all $x \in T$. Then $(f,a)(f,a)^*(f,a) = (f,a)(g,a^*)(f,a) = (fg_a, aa^*)(f,a) = (fg_af_{aa^*}, aa^*a) = (fg_af_{aa^*}, a)$, where $(fg_af_{aa^*})(x) = f(x)g(xa)f(xaa^*) = f(x)f(xaa^*)^*f(xaa^*) = f(x)f(x)^*f(x) = f(x)$ for all $x \in T$. Hence $(f,a)(f,a)^*(f,a) = (f,a)$ for all $(f,a) \in S \odot T$. Therefore $S \odot T = S^T \times T$ is a regular $*$ - semigroup.

**Theorem 4.5**. *Let $S$ be a topological regular $*$ - semigroup and let $T$ be a locally compact topological regular $*$ - semigroup. Suppose that the semigroup $S^T$ of all continuous fucntions from $T$ into $S$ is locally compact and suppose $\phi : T \to End(S^T)$ given by $\phi(a)(f) = f \circ \rho_a$ is continuous. If $f(xe) = f(x)$ for all $x \in T$ and $f \in S^T$ , then the wreath product $S \odot T$ of $S$ and $T$ is a topological regular $*$ - semigroup.*

*Proof.* In view of Theorem 4.3.,$S \odot T$ is a topological semigroup. In view of Theorem 4.4., $S \odot T$ is a regular $*$ - semigroup. We need to show that the unary operation on $S \odot T = S^T \times T$ is continuous. To prove this, we adopt the following notations;

(1) $Uni_{S^T}$ and $Uni_T$ are unary operations on $S^T$ and $T$ respectively,
(2) $\pi_1 : S^T \times T \to S^T$ is the first projection, and
(3) $\pi_2 : S^T \times T \to T$ is the second projection. Then the unary operation on $S \odot T$ is $(Uni_{S^T} \circ \phi(a^*) \circ \pi_1) \times (Uni_T \circ \pi_2)$. Hence it is continuous. Therefore $S \odot T$ is a topological regular * - semigroup.

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