1. Introduction

Let $K$ be a field and $R$ be a $K$-algebra. If $R$ is finitely generated with a finite dimensional generating $K$-subspace $V$ containing 1 (that is, $R = \bigcup_{n=1}^{\infty} V^n$), then the real number, $\limsup \log_n(\dim_K V^n)$, is independent of the generating subspace $V$ of $R$. We call this number the Gelfand-Kirillov dimension of $R$ and write

$$GK\dim(R) = \limsup \log_n(\dim_K V^n).$$

For an infinitely generated $K$-algebra $R$, the Gelfand-Kirillov dimension is defined by

$$GK\dim(R) = \sup S \{GK\dim(S)\}$$

where $S$ ranges over finitely generated subalgebras of $R$.

For a left $R$-module $M$, the Gelfand-Kirillov dimension of $M$ is given by

$$GK\dim_R(M) = \sup_{V,N} \{\limsup \log_n(\dim_K (V^n N))\}$$

where the supremum is taken over all finite dimensional subspace $V$ of $R$ containing 1 and all finite dimensional subspace $N$ of $M$. The Gelfand-Kirillov dimension of a right $R$-module is defined similarly.

It is known that if $R \subseteq S$ are $K$-algebras such that $S_R$ is finitely generated as a right $R$-module then $GK\dim_S(S \otimes_R M) = GK\dim_R(M)$ for all left $R$-module $M$. But this property is not left-right symmetric. In this article, we construct algebras $R \subseteq S$ and a left $R$-module $M$ such that $R_S$ is finitely generated as a left $R$-module with $GK\dim_S(S \otimes_R M) \neq GK\dim_R(M)$. Also we will prove that if $R$ is an algebra with finite Gelfand-Kirillov dimension and if $GK\dim_R(M) = GK\dim(M)$ for all finitely generated left $R$-module $M$, then $R$ is strongly $\tau$-regular.

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**LEMMA 1.** Let $R$ be a $K$-algebra and $M$ be a left $R$-module. Then

1. $GK\dim_R(M) \leq GK\dim(R)$,
2. If $M$ is finitely generated and $\theta : M \to M$ is a one-to-one homomorphism, then
   $$GK\dim_R(M/\theta(M)) \leq GK\dim_R(M) - 1,$$
3. If $M = M_1 + M_2 + \ldots + M_n$ is a sum of submodules $M_i$, then
   $$GK\dim_R(M) = \max\{GK\dim_R(M_i) | i = 1, 2, \ldots, n\}.$$

**Proof.** [2, Proposition 5.1.]

**LEMMA 2.** Let $R \subset S$ be $K$-algebras such that $RS$ is finitely generated as a left $R$-module. Then for every right (or left) $S$-module $M$

$$GK\dim_R(M) = GK\dim_S(M)$$

**Proof.** If $M$ is a right module, then the result follows from [1, Corollary 6 (ii)]. We only need to prove the lemma when $M$ is a left module. Clearly we have $GK\dim_R(M) \leq GK\dim_S(M)$.

Suppose $S = Rs_1 + \ldots + Rs_m$ ($m \geq 1, s_i \in S$). Let $V$ be a finite dimensional subspace of $S$ with a $K$-basis $\{v_1, v_2, \ldots, v_p\}$ and let $X$ be a finite dimensional subspace of $M$ with a $K$-basis $\{x_1, x_2, \ldots, x_q\}$. By the definition of Gelfand-Kirillov dimension, we may assume that $s_k \in V$ for all $k$. Set $u_i = \sum_{k=1}^m r_{ik} s_k$ and $v_i v_j = \sum_{k=1}^m r_{ijk} s_k$, where $r_{ik}, r_{ijk} \in R$ ($1 \leq i, j \leq p, 1 \leq k \leq m$). Let $U$ be the subspace of $R$ spanned by $\{r_{ik}, r_{ijk}\}$ and $Y$ be the subspace of $M$ spanned by $\{s_k x_l | 1 \leq k \leq m, 1 \leq l \leq q\}$. Then $U$ and $Y$ are finite dimensional subspaces of $R$ and $M$ respectively. By induction, we have $V^n \subset \sum_{k=1}^m U^{2^n} s_k$ and hence

$$V^n X \subset (\sum_{k=1}^m U^{2^n} s_k) X = \sum_{k=1}^m \sum_{l=1}^q U^{2^n} s_k x_l = U^{2^n} Y$$

for all $n \geq 1$. Thus $\dim_K(V^n X) \leq \dim_K(U^{2^n} Y)$ and

$$\log_n(\dim_K(V^n X)) \leq \log_n(\dim_K(U^{2^n} Y)) \leq GK\dim_R(M).$$

Therefore $GK\dim_S(M) \leq GK\dim_R(M)$. 
Corollary 3. Let \( R \subseteq S \) be \( K \)-algebras. If \( S \) is finitely generated as a right (or left) \( R \)-module, then \( \text{GK dim}(R) = \text{GK dim}(S) \).

2. A counter example

Proposition 4. Let \( R \subseteq S \) be \( K \)-algebras such that \( S_R \) is finitely generated. Then for each left \( R \)-module \( M \), \( \text{GK dim}_S(S \otimes_R M) = \text{GK dim}_R(M) \).

Proof. Since \( S_R \) is finitely generated, by Lemma 2 it follows that

\[
\text{GK dim}_S(S \otimes_R M) = \text{GK dim}_R(S \otimes_R M) \\
\geq \text{GK dim}_R(R \otimes_R M) = \text{GK dim}_R(M).
\]

On the other hand, by [1, Corollary 6(i)],

\[
\text{GK dim}_S(S \otimes_R M) \geq \text{GK dim}_R(M).
\]

For the case when \( R \subseteq S \) is finitely generated, Proposition 4 may not be true. We have the following example.

Example 5. Let \( K \langle x, y \rangle \) be the noncommuting free algebra in two indeterminates \( x \) and \( y \), and \( I \) be the ideal generated by \( yx \) and \( y^2 \). Let \( S = K \langle x, y \rangle / I \) and \( R = K \langle x \rangle \) the polynomial ring in \( x \). Then \( R \subseteq S \) such that \( S = R \oplus R_y \) as left \( R \)-modules. So \( R \subseteq S \) is finitely generated (but \( S_R \) is not finitely generated). According to the Corollary 3, we have \( \text{GK dim}(S) = \text{GK dim}(R) = 1 \).

Let \( J = Rx = K \langle x \rangle x \) be the ideal of \( R \) generated by \( x \) and \( R \subseteq R = R/J \cong_R K \) (here the \( R \)-action on \( K \) is given by \( f(x) \cdot \alpha = f(0) \alpha \) for all \( f(x) \in R, \alpha \in K \)).

Since \( S = R \oplus R_y \) as \( (R, R) \)-bimodules, \( R_y \otimes_R M \) is a submodule of \( R \otimes_R M \), and hence

\[
\text{GK dim}_S(S \otimes_R M) = \text{GK dim}_R(S \otimes_R M) \geq \text{GK dim}_R(R_y \otimes_R M).
\]

On the other hand, \( (R_y)J = 0 \) and \( J \subseteq R_y \) imply that

\[
R_y \otimes_R M \cong_R R_y \otimes_R / J \cong_R R_y \otimes_R K \cong R_y
\]
as left $R$-modules. Thus
\[ \text{GKdim}_R(Ry \otimes_R M) = \text{GKdim}_R(Ry) = \text{GKdim}(R) = 1 \]
and
\[ \text{GKdim}_S(S \otimes_R M) \geq \text{GKdim}_R(Ry \otimes_R M) = 1. \]
But $\text{GKdim}_R(M) = \text{GKdim}_R(K) = 0$, therefore $\text{GKdim}_R(M) \neq \text{GKdim}_S(S \otimes_R M)$.

3. Algebras whose nonzero modules have the same Gelfand-Kirillov dimension

In this section we investigate the class of algebras whose nonzero modules have the same Gelfand-Kirillov dimension. For example, if $R$ is an algebra finitely generated as a left module over a locally finite algebra, then $R$ has this property. Indeed $\text{GKdim}_R(M) = 0$ for every nonzero left $R$-module $M$. Another example is following. If $R$ is an algebra finitely generated as a left module over a simple Artinian algebra, then for every nonzero left $R$-module $M$, $\text{GKdim}_R(M) = \text{GKdim}(R)$. The following Theorem asserts that such an algebra is strongly $\pi$-regular.

\textit{Definition}. A ring $R$ is said to be strongly $\pi$-regular if for each $x \in R$ there exist a positive integer $n$ and an element $y \in R$ depending on $x$ such that $x^n = yx^{n+1}$.

\textbf{Theorem 6}. Let $R$ be a $K$-algebra such that $\text{GKdim}(R) < \infty$. If $\text{GKdim}_R(M) = \text{GKdim}(R)$ for every finitely generated left $R$-module $R M$, then $R$ is strongly $\pi$-regular.

\textit{Proof}. Let $x \in R$ be a non-nilpotent element. Set
\[ I = \{ r \in R | rx^n = 0 \text{ for some } n \geq 1 \} \]
then $I$ is a proper left ideal since $x \notin I$. Furthermore for $r \in R$, $r \in I$ if and only if $rx^n \in I$ for all $n \geq 1$. Thus the mapping,
\[ \theta : R/I \rightarrow (Rx^n + I)/I \]
given by $\theta(r + I) = rx^n + I$ ($r \in R$), is a well-defined isomorphism of left $R$-modules. By Lemma 1(2),

$$GK\dim_R(R/(Rx^n + I)) = GK\dim_R((R/I)/\theta(R/I))$$

$$\leq GK\dim_R(R/I) - 1.$$ 

According to the assumption, $R/(Rx^n + I) = 0$ or equivalently $R = Rx^n + I$ for all $n \geq 1$. In particular, $R = Rx + I$. Since $1 \in R$, $1 = yx + a$ ($y \in R, a \in I$), and since $ax^n = 0$ for some $n \geq 1$, we have

$$x^n = yx^{n+1} + ax^n = yx^{n+1}. $$

References


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