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GELFAND–KIRILLOV DIMENSION OF MODULES

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1. Introduction

Let K be a field and R be a K-algebra. If R is finitely generated with a finite dimensional generating K-subspace V containing 1 (that is, $R = \cup_{n=1}^{\infty} V^n$), then the real number, $\limsup(\log_n(\dim_K V^n))$, is independent of the generating subspace V of R. We call this number the *Gelfand-Kirillov dimension* of R and write

$$GK\dim(R) = \limsup(\log_n(\dim_K V^n)).$$

For an infinitely generated K-algebra R, the Gelfand-Kirillov dimension is defined by

$$GK\dim(R) = \sup_S\{GK\dim(S)\}$$

where S ranges over finitely generated subalgebras of R.

For a left R-module M, the Gelfand-Kirillov dimension of M is given by

$$GK\dim_R(M) = \sup_{V,N}\{\limsup(\log_n(\dim_K(V^n N)))\}$$

where the supremum is taken over all finite dimensional subspace V of R containing 1 and all finite dimensional subspace N of M. The Gelfand-Kirillov dimension of a right R-module is defined similarly.

It is known that if $R \subset S$ are K-algebras such that S_R is finitely generated as a right R-module then $GK\dim_S(S \otimes_R M) = GK\dim_R(M)$ for all left R-module M. But this property is not left-right symmetric. In this article, we construct algebras $R \subset S$ and a left R-module M such that ${}_RS$ is finitely generated as a left R-module with $GK\dim_S(S \otimes_R M) \neq GK\dim_R(M)$. Also we will prove that if R is an algebra with finite Gelfand-Kirillov dimension and if $GK\dim_R(M) = GK\dim(R)$ for all finitely generated left R-module M, then R is strongly π-regular.

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LEMMA 1. *Let R be a K-algebra and M be a left R-module. Then*

(1) $GK\dim_R(M) \leq GK\dim(R)$,

(2) *If M is finitely generated and $\theta : M \longrightarrow M$ is a one-to-one homomorphism, then*

$$GK\dim_R(M/\theta(M)) \leq GK\dim_R(M) - 1,$$

(3) *If $M = M_1 + M_2 + ... + M_n$ is a sum of submodules M_i, then*

$$GK\dim_R(M) = \max\{GK\dim_R(M_i) | i = 1, 2, ..., n\}.$$

Proof. [2, Proposition 5.1.]

LEMMA 2. *Let $R \subset S$ be K-algebras such that ${}_RS$ is finitely generated as a left R-module. Then for every right (or left) S-module M*

$$GK\dim_R(M) = GK\dim_S(M)$$

Proof. If M is a right module, then the result follows from [1, Corollary 6 (ii)]. We only need to prove the lemma when M is a left module. Clearly we have $GK\dim_R(M) \leq GK\dim_S(M)$.

Suppose $S = Rs_1 + ... + Rs_m$ $(m \geq 1, s_i \in S)$. Let V be a finite dimensional subspace of S with a K-basis $\{v_1, v_2, ..., v_p\}$ and let X be a finite dimensional subspace of M with a K-basis $\{x_1, x_2, ..., x_q\}$. By the definition of Gelfand-Kirillov dimension, we may assume that $s_k \in V$ for all k. Set $v_i = \sum_{k=1}^{m} r_{ik} s_k$ and $v_i v_j = \sum_{k=1}^{m} r_{ijk} s_k$, where $r_{ik}, r_{ijk} \in R$ $(1 \leq i, j \leq p, 1 \leq k \leq m)$. Let U be the subspace of R spanned by $\{r_{ik}, r_{ijk}\}$ and Y be the subspace of M spanned by $\{s_k x_l | 1 \leq k \leq m, 1 \leq l \leq q\}$. Then U and Y are finite dimensional subspaces of R and M respectively. By induction, we have $V^n \subset \sum_{k=1}^{m} U^{2n} s_k$ and hence

$$V^n X \subset (\sum_{k=1}^{m} U^{2n} s_k) X = \sum_{k=1}^{m} \sum_{l=1}^{q} U^{2n} s_k x_l = U^{2n} Y$$

for all $n \geq 1$. Thus $\dim_K(V^n X) \leq \dim_K(U^{2n} Y)$ and

$$\log_n(\dim_K(V^n X)) \leq \log_n(\dim_K(U^{2n} Y)) \leq GK\dim_R(M).$$

Therefore $GK\dim_S(M) \leq GK\dim_R(M)$.

COROLLARY 3. *Let $R \subset S$ be K-algebras. If S is finitely generated as a right (or left) R-module, then $GK\dim(R) = GK\dim(S)$.*

2. A counter example

PROPOSITION 4. *Let $R \subset S$ be K-algebras such that S_R is finitely generated. Then for each left R-module M, $GK\dim_S(S \otimes_R M) = GK \dim_R(M)$.*

Proof. Since S_R is finitely generated, by Lemma 2 it follows that

$$\begin{aligned} GK\dim_S(S \otimes_R M) &= GK\dim_R(S \otimes_R M) \\ &\geq GK\dim_R(R \otimes_R M) = GK\dim_R(M). \end{aligned}$$

On the other hand, by [1, Corollary 6(i)],

$$GK\dim_S(S \otimes_R M) \leq GK\dim_R(M).$$

For the case when $_RS$ is finitely generated, Proposition 4 may not be true. We have the following example.

Example 5. Let $K\langle x, y\rangle$ be the noncommuting free algebra in two indeterminates x and y, and I be the ideal generated by yx and y^2. Let $S = K\langle x, y\rangle / I$ and $R = K[x]$ the polynomial ring in x. Then $R \subset S$ such that $S = R \oplus Ry$ as left R-modules So $_RS$ is finitely generated (but S_R is not finitely generated). According to the Corollary 3, we have $GK\dim(S) = GK\dim(R) = 1$.

Let $J = Rx = K[x]x$ be the ideal of R generated by x and $_RM = R/J \cong {}_RK$ (here the R-action on K is given by $f(x) \cdot \alpha = f(0)\alpha$ for all $f(x) \in R$, $\alpha \in K$).

Since $S = R \oplus Ry$ as (R, R)-bimodules, $Ry \otimes_R M$ is a submodule of $_RS \otimes_R M$, and hence

$$GK\dim_S(S \otimes_R M) = GK\dim_R(S \otimes_R M) \geq GK\dim_R(Ry \otimes_R M).$$

On the other hand, $(Ry)J = 0$ and $JM = 0$ imply that

$$Ry \otimes_R M \cong Ry \otimes_{R/J} M \cong Ry \otimes_K K \cong Ry$$

as left R-modules. Thus

$$GK\dim_R(Ry\otimes_R M) = GK\dim_R(Ry) = GK\dim(R) = 1$$

and

$$GK\dim_S(S\otimes_R M)\geq GK\dim_R(Ry\otimes_R M) = 1.$$

But $GK\dim_R(M) = GK\dim_R(K) = 0$, therefore $GK\dim_R(M)\neq GK \dim_S(S\otimes_R M)$.

3. Algebras whose nonzero modules have the same Gelfand-Kirillov dimension

In this section we investigate the class of algebras whose nonzero modules have the same Gelfand-Kirillov dimension. For example, if R is an algebra finitely generated as a left module over a locally finite algebra, then R has this property. Indeed $GK\dim_R(M) = 0$ for every nonzero left R-module M. Another example is following. If R is an algebra finitely generated as a left module over a simple Artinian algebra, then for every nonzero left R-module M, $GK\dim_R(M) = GK\dim(R)$. The following Theorem asserts that such an algebra is strongly π-regular.

Definition. A ring R is said to be strongly π-regular if for each $x\in R$ there exist a positive integer n and an element $y\in R$ depending on x such that $x^n = yx^{n+1}$.

THEOREM 6. *Let R be a K-algebra such that $GK\dim(R) < \infty$. If $GK\dim_R(M) = GK\dim(R)$ for every finitely generated left R-module ${}_RM$, then R is strongly π-regular.*

Proof. Let $x \in R$ be a non-nilpotent element. Set

$$I = \{r \in R | rx^n = 0 \text{ for some } n \geq 1\}$$

then I is a proper left ideal since $x \notin I$. Furthermore for $r \in R$, $r \in I$ if and only if $rx^n \in I$ for all $n \geq 1$. Thus the mapping,

$$\theta : R/I \longrightarrow (Rx^n + I)/I$$

given by $\theta(r+I) = rx^n + I$ $(r \in R)$, is a well-defined isomorphism of left R-modules. By Lemma 1(2),

$$\begin{aligned} GK\dim_R(R/(Rx^n+I)) &= GK\dim_R((R/I)/\theta(R/I)) \\ &\leq GK\dim_R(R/I) - 1. \end{aligned}$$

According to the assumption, $R/(Rx^n+I) = 0$ or equivalently $R = Rx^n + I$ for all $n \geq 1$. In particular, $R = Rx + I$. Since $1 \in R$, $1 = yx + a$ $(y \in R,\ a \in I)$, and since $ax^n = 0$ for some $n \geq 1$, we have

$$x^n = yx^{n+1} + ax^n = yx^{n+1}.$$

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