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ON EPIC AND MONIC ENDOMORPHISMS

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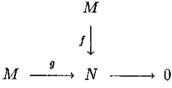
0. Introduction

Assume that ring R is an associtive ring with an identity and every module $M = {}_{R}M$ is a left R-module.

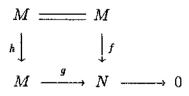
The ring of all *R*-endomorphisms on a left *R*-module *M*, denoted by End(M) will be written on the right side of *M* as right operators on *M*, that is, $_{R}M_{End(M)}$ will be considered on this paper. For *R*homomorphisms $f : L \to M$, $g : M \to N$, their composition fg : $L \xrightarrow{f} M \xrightarrow{g} N$ of f and g is written in the arrow orders of f and g, for any left *R*-modules L, M and N.

According to [5], a module $_{R}M$ is quasi-projective in case it is Mprojective (or, projective relative to M itself). This definition is equivalent to the following definition of [1](see 16.7 Proposition p148,[5]). In other words, we can replace N with M/T for each submodule T of M.

DEFINITION 1. An R-module M is said to be quasi-projective if every diagram



where the bottom row is exact with f,g are R-homomorphisms, has a commutative diagram;



there exists an endomrophism h on M such that f = hg.

Dedicated to Professor Younki Chae on his 61st birthday Received September 9,1993.

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THEOREM 2. [3] Let M be a (quasi-)projective left R-module. Then $Rad(End(M)) = \{f \in End(M) | Imf \text{ is superfluous (or small) in } M\}.$

Let's use the notation E(M) to stand for an *injective hull* of M.

THEOREM 3. [6, p49] Let M be a (quasi-)injective left R-module. Then,

 $Rad(End(M)) = \{f \in End(M) | kerf \text{ is essential (or large) in } M\}.$

DEFINITION 4. ([6], p 48) A module M is said to be quasi-injective provided the natural map $Hom_R(M, M) \rightarrow Hom_R(L, M)$ is a surjective for all $L \leq M$, i.e., provided any homomorphism from a submodule of M into M extends to an endomorphism of M.

1. Results

In order to reveal the relationships between submodules of a module and ideals of its endomorphism rings, firstly we must find the structures of the radical of the endomorphism whose element is left-(or right)quasi-regular. To do this, we need to see whether endomorphisms are left invertible or right invertible, or not. Notice the composition of maps follows arrow- direction. To make a comparison of (quasi-)projectivity with (quasi-)injectivity, I will write some results from [10] with respect to (quasi-)injectivity.

THEOREM 1. Let a left R-module $_RM$, be (quasi-)projective and let f be an endomorphism in $End(_RM)$. Then we have f is an epimorphism if, and only if, f has a left inverse.

Proof. "If" part is easy. Now let's prove the "only if" part. Let $f: M \to M$ be an epimorphism. Then we have the induced isomorphism $\overline{f}: M/\ker f \to M$ by the first isomorphism theorem. Consider the following diagram with the natural map n;

$$M = M$$

$$h \downarrow \qquad \qquad \qquad \downarrow \overline{f}^{-1}$$

$$M \xrightarrow{n} M/kerf \longrightarrow 0$$

and then we have an R-endomorphism $h: M \to M$ such that $\overline{f}^{-1} = hn$, from which we also have $hn\overline{f} = 1_M$. Actually $hn\overline{f} = hf = 1_M$ which proves the theorem.

THEOREM 2. For a left (quasi-)injective R-module $_{R}M$, and an endomorphism f, f is a monomorphism if, and only if, f has a right inverse.

Proof. " If " part is easy. Now let's prove the "only if " part. Since every module $_{R}M$ has an *injective envelope* $E(_{R}M) = E(M)$, we need to show that each monomorphism $f: 0 \to M \xrightarrow{f} M$, there exists an *R*-module homomorphism $g: M \to M$ such that $fg = 1_M$. From the the definition of injective envelope E(M), for a monomorphism $fi: M \to E(M)$, there is an extension $\overline{f}: E(M) \to E(M)$ such that $\overline{f}_{M} = fi$. Now that there exists a $\overline{g}: E(M) \to E(M)$ such that $\overline{f}_{\overline{g}} = 1_{E(M)}$, $Im \overline{f}$ is a direct summand of E(M).

Let's put $g = \overline{g}|_M = ig : M \subset E(M) \to E(M)$, then for each $x \in M$, $xfg = xfig = x\overline{f}g = x\overline{f}\overline{g} = x1_{E(M)} = x$. Thus $fg = 1_M$.

REMARK 3. The condition "(quasi-)projectivity" of M in the above theorem 1 is necessary for any epimorphism having a left inverse in the endomorphism ring of M. For an example, take R = Z, $_RM = Z(p^{\infty})$, the multiplication by p,in fact, it is an epimorphism in $End(_RM)$. We now have an epimorphism having no left inverse in End(M). Without any hesitation we have that $Z(p^{\infty})$ is not a (quasi-)projective Z-module.

In the above theorem 2, there is no guarantee for existing the inverse g of a monomorphism f in End(M). For an example, take R = Z, $_{R}M = Z$, $f = \times 2$, the multiplication by 2, then Z has its envelope Q, f has its right inverse $g = \div 2$ the division by 2 whose range $\{\frac{1}{2}s \mid s \in Z\}$ is not contained in $Z = _{R}M$. However f has g as its inverse homomorphism in $Hom_{R}(Imf, Z)$ or $Hom_{R}(Z, Q)$. Immediately, we conclude that Z is not (quasi-)injective.

Here we can add one more equivalent condition for an endomorphism f to be an epimorphism to the Proposition 3.4. p44 in [5].

PROPOSITION 4. For any (quasi-)projective left R-module M and $f: M \to M$ an endomorphism the followings are equivalent:

(a) f is an epimorphism .

- (b) $\operatorname{Im}(f) = M$.
- (c) For every $_{R}K$ and every pair $g, h : M \to K$ of R homomorphisms, fg = fh implies g = h.
- (d) For every $_{R}K$ and every R-homomorphism $g: M \to K$, fg = 0 implies g = 0.
- (e) f has a left inverse, in fact, in $End_R(M)$.

Proof. Proof is immediately followed from Theorem 1.

Similarly, we can add one more equivalent condition of an endomorphism f to be a monomorphism to the proposition 3.4. p 44 in [5]

PROPOSITION 5. For any left (quasi-)injectiveR-module M and an endomorphism $f: M \to M$, the followings are equivalent:

- (a) f is a monomorphism.
- (b) ker $f = \{0\}$.
- (c) For every $_{R}K$ and every pair $g, h : K \to M$ of R homomorphisms, gf = hf implies g = h.
- (d) For every $_{R}K$ and every R-homomorphism $g: K \to M, gf = 0$ implies g = 0.
- (e) f has a right inverse. (In fact, in $Hom_R(Imf, M)$ or $Hom_R(M, E(M))$, where E(M) is an injective envelope of M.)

Proof. Proof is immediately followed from Theorem 2.

The following corollaries are followed easily, and thus we omit those proofs.

COROLLARY 6. In the endomorphism ring $End(_{\mathbb{R}}M)$ of any (quasi -)projective left R-module $_{\mathbb{R}}M$, if the composition gf is an epimorphism with $f, g \in End(_{\mathbb{R}}M)$, then so is f.

COROLLARY 7. In the endomorphism ring $End(_{\mathbb{R}}M)$ of any left (quasi-)injective R-module $_{\mathbb{R}}M$, if the composition fg is a monomorphism, so is f.

LEMMA 8. If $_{R}M$ is (quasi-)projective. Then no epimorphism in $End(_{R}M)$ is contained in a left proper ideal of $End(_{R}M)$.

Proof. Suppose that a proper left ideal I of $End(_{R}M)$ contains an epimorphism f. Since $_{R}M$ is (quasi-)projective, by theorem 1, there

exists an endomorphism g in $End(_{R}M)$ such that $gf = 1_{M}$ which implies that $I = End(_{R}M)$. It contradicts to be a proper left ideal *I*. Hence we completes the proof.

LEMMA 9. If $_{R}M$ is (quasi-)injective. Then no monomorphism in $End(_{R}M)$ is contained in a right proper ideal of $End(_{R}N)$.

Proof. Proof is easily followed by the same way of the above corollary.

Now, it is worth considering left ideals of End(M) of the form

$$I^{N} = Hom_{R}(M, N) = \{ f \in End(M) \mid Imf \subseteq N \},\$$

for a submodule N of M. and right ideals of End(M) of the form

$$I_N = \{ f \in End(M) \, | \, N \subseteq ker \, f \},\$$

for a submodule N of M.

THEOREM 10. For a (quasi-)projective module $_{\mathbb{R}}M$ and any small(or superfluous) submodule N of M, the right ideal I^{N} is small in End(M).

Proof. We need only consider all left ideals of End(M). Suppose that J is a left ideal of End(M) such that $I^N + J = End(M)$. Then the identity 1_M can be written as a sum of $f \in I^N$ and $j \in J$, i.e., $1_M = f + j$. Then $M = Im1_M = Im(f+j) \leq Imf + Imj \leq N + Imj$, which implies that N + Imj = M. Since N is small in M, we have Imj = M, saying that j is an epimorphism. By Theorem 1, j has a left inverse in End(M), hence J = End(M). Therefore we have proved I^N is small in End(M).

THEOREM 11. For a (quasi-)injective module $_{\mathbb{R}}M$ and any large submodule N of M, the right ideal I_N is small in End(M).

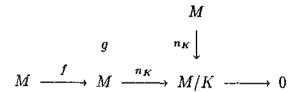
Proof. We need only consider all right ideals of End(M). Suppose that J is a right ideal of End(M) such that $I_N + J = End(M)$. Then the identity 1_M can be written as a sum of $f \in I_N$ and $j \in J$, i.e., $1_M = f + j$. Then $0 = ker1 = ker(f + j) \ge kerf \cap kerj \ge N \cap kerj$, which implies that $N \cap kerj = 0$. Since N is large in M, we have kerj = 0, saying that j is a monomorphism. By Theorem 2, j has a right inverse in End(M), hence J = End(M). Therefore we have proved I_N is small in End(M).

Here is an easier proof than that had done in [1] and a smaller method of proof in [5] of the following fact restated Lemma 1 in [3] in using our tool I^N .

LEMMA 12. ([3]) Let M be a (quasi-)projective left R-module. Then,

 $Rad(End(M)) = \{f \in End(M) | Imf \text{ is superfluous (or small) in } \forall \}$

Proof. Suppose that an endomorphism f has a small image fn fin M. Considering the left ideal $I^{Imf} = \{h \in End(M) | Imh \subseteq Imf\}$, we have a small left ideal I^{Imf} of End(M) by Theorem 6, which contains f. Since the radical Rad(End(M)) is the largest small left ideal of $End(M), f \in I^{Imf} \subseteq Rad(End(A))$. Conversely let $f \gg 10$ Rad(End(M)) and let K be a submodule of M such that Imf+K = M.



where n_K is the natural epimorphism. Since fn_K is an epimorphism, there exists an endomorphism $g: M \to M$ such that $gfn_K = n_K$. Thus we have $(1 - gf)n_K = 0$. Since $f \in Rad(End(M), 1 - gf)$ is invertible, and so $n_K = 0$. Hence K = M which completes the proof.

REMARK 13. According to the propositions ([4],p118) and [8], the radical Rad(M), the socle Soc(M) of M and the relations are;

$$Rad M = \bigcap \{K < M | K \text{ is maximal in} M\}$$
$$= \sum \{L < M | L \text{ is small in } M\},$$
$$\sum_{a \in A} I^{N_a} \le I^{\sum_{a \in A} N_a} \text{ for every } N_a \le M,$$

Soc
$$M = \sum \{K < M \mid K \text{ is minimal in } M\}$$

= $\bigcap \{L < M \mid L \text{ is large in } M\},$
and $\sum_{a \in A} I_{N_a} \leq I_{\bigcap_{a \in A} N_a} \text{ for every } N_a \leq M.$

THEOREM 14. For any left (quasi-)projective module M with $Rad M = \sum_{a \in A} N_a$, small submodule N_a of M, we have

$$\sum_{a \in A} I^{N_a} = Rad(End(M)).$$

Proof. Since Rad(End(M)) is the unique largest small left ideal of End(M), thus for every small submodule N_a of M, the small left ideal I^{N_a} is contained in the radical Rad(End(M)) of End(M) for every $a \in A$. Hence the sum $\sum_{a \in A} I^{N_a}$ is also contained in Rad(End(M)). Conversely, assume f is in Rad(End(M)), Imf is small in M by Theorem 8. And I^{Imf} is a small left ideal of End(M), $f \in I^{Imf} \leq \sum_{a \in A} I^{N_a}$ for every small submodule N_a of M. Thus we complete the proof.

THEOREM 15. For any left (quasi-)injective module M with Soc $M = \bigcap_{a \in A} N_a$, large submodule N_a of M, we have

$$\sum_{a \in A} I_{N_a} = Rad(End(M)).$$

Proof. From Theorem in [5], Rad(End(M)) is the unique largest small right ideal of End(M), and thus for every large submodule N_a of M, the small right ideal I_{N_a} is contained in the radical Rad(End(M)) of End(M) for every $a \in A$. Hence the sum $\sum_{a \in A} I_{N_a}$ is also contained in Rad(End(M)).

Conversely, assume f is in Rad(End(M)), kerf is large in M by Theorem 3 in the introduction. And I_{kerf} is a small right ideal of End(M), $f \in I_{kerf} \leq \sum_{a \in A} I_{N_a}$ for every large submodule N_a of M. Thus we complete the proof.

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REMARK 16. We summarize these two theorems 14,15, then we conclude;

 In a (quasi-)projective module M, if the radical Rad(M) of M is small,

easily we can conclude that $Rad(End(M)) = I^{Rad(M)}$.

- (2) In a (quasi-)injective module M with large socle soc(M), we obtain $Rad(End(M)) = I_{soc(M)}$.
- (3) Moreover in a left (quasi-)projective and (quasi-) injective module M with (1), or (2) condition, we have $Rad(End(M) = I^{Rad(M)}$, or $I_{soc(M)}$, respectively.

The following corollaries are followed immediately.

COROLLARY 17. For any left (quasi-)projective module M, we have the followings:

- (a) If Rad(End(M)) = 0, then there is no non-zero endomorphism whose image is small in M;
- (b) $Rad(End(M)) \leq I^{Rad(M)}$;
- (c) If Rad(M) = 0, then Rad(End(M)) = 0.

COROLLARY 18. For any left (quasi-)injective module M, we have the followings :

- (a) If Rad(End(M)) = 0, then there is no non-zero endomorphism whose kernel is large in M;
- (b) $Rad(End(M)) \leq I_{Soc(M)}$;
- (c) If Soc(M) = M, then Rad(End(M)) = 0.

EXAMPLES 19. Since it is well-known that $M = Z(2^{\infty})$ is injective, and $\{\overline{0}, \overline{(1/2)}\}$ is the socle of M, and it is essential (i.e., large) in M, and thus End(M) has the radical $I_{soc(M)}$ isomorphic to 2Z.

In the second example, when R = Z = M, M = Z has zero radical. Since M is a free, (quasi-) projective module with Rad(M) = 0, its endomorphism ring End(M) has radical $I^{Rad(M)} = I^0 = 0$.

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