Pusan Kyŏngnam Math. J. 9(1993), No. 2, pp 251-260

ON THE PROPERTIES OF JOINTLY OPERATORS

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1. Introduction

Throughout this paper, H will a complex Hilbert space and all operators on H will be assumed to be linear and bounded.

$B(H)^n$ will denote the set of all n-tuple of operators $T = (T_1, T_2, \ldots, T_n)$. For $T = (T_1, T_2, \ldots, T_n) \in B(H)^n$ and $x \in H$,we denote

$$\|Tx\| = (\sum_{i=1}^{n} \|T_i x\|^2)^{\frac{1}{2}},$$

$$|(Tx, x)| = (\sum_{i=1}^{n} |(T_i x, x)|^2)^{\frac{1}{2}}.$$

Let $\sigma_\pi(T)$ denote its approximate point spectrum, $\sigma_l(T)$ its left spectrum , $\sigma_r(T)$ its right spectrum, $\sigma_H(T)$ its Harte spectrum, $\sigma'(T)$ its commutant spectrum, $\sigma''(T)$ its double commutant spectrum, $\sigma_T(T)$ its Taylor spectrum, $\sigma_P(T)$ its polynomial spectrum, $W(T)$ its joint numerical range and $w(T)$ its joint numerical radius.

The joint operator norm and joint spectral radius of T is defined by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|,$$

$$r_\bullet(T) = \sup\{|\lambda| : \sigma_\bullet(T)\},$$

where $\sigma_\bullet = \sigma_\pi, \sigma_l, \sigma_r \sigma_H, \sigma', \sigma'', \sigma_T$ or σ_P, and $r_\bullet = r_\pi, r_l, r_H, r', r'', r_T$ or r_P. Let we denote $\sigma = \sigma_\pi, \sigma_l, \sigma_H$ or σ_T and $r = r_\pi, r_l, r_H$ or r_T.

The maximal joint numerical range of T is defined by the set $W_0(T) = \{\lambda : ((T_1 x_n, x_n), (T_2 x_n, x_n), \ldots, (T_n x_n, x_n)) \to \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \|x\| = 1 \text{and} |Tx_n\| \to \|T\|\}$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in C^n$.

Received September 21, 1993.

* This research was supported by the Ministry of ducation,1991 .

We shall call T jointly normaloid if $\|T\| = r(T)$ and call T jointly transloid if $T - \lambda = (T_1 - \lambda_1, T_2 - \lambda_2, \dots, T_n - \lambda_n)$ is jointly normaloid for any point $\lambda \in C^n$.

Any n–tuple of commuting operators $T = (T_1, T_2, \dots, T_n)$ on a Hilbert space satisfies the inequality

$$\sup_{\|x\|=1} \{\|Tx\|^2 - |(Tx,x)|^2\} \geq R_T^2$$

where R_T is the radius of the smallest ball containing the Taylor spectrum of T. Moroever, if T is jointly transloid, then the equality holds. (Fan Ming [1])

We shall define the jointly transcendental radius M_T of $T = (T_1, T_2, \dots, T_n)$ as $M_T = B_T^2$ where

$$B_T = \sup_{\|x\|=1} \{\|Tx\|^2 - |(Tx,x)|^2\}.$$

In Takaguchi [2], the center of mass for an n–tuple of operators has been defined and stated that the center of mass of T is coincident with the center of the smallest sphere containing the joint spectrum of T in case of a jointly transloid n–tuple $T = (T_1, T_2, \dots, T_n)$ of operators.

In this note we shall define the jointly centroid operator with the center of mass for an n–tuple of operators and state the properties of this class of operators.

With the properties of jointly centroid operators we can find a new inclusion relation between jointly centroid operators and jointly transloid and show that the center of mass of jointly centroid operator is coincident with the center of the smallest closed ball containing $\sigma(T)$.

2. Center of mass for an n–tuple of operators $T = (T_1, T_2, \dots, T_n)$

THEOREM 1 (TAKAGUCHI [2]). *For an n–tuple $T = (T_1, T_2, \dots, T_n)$ of operators the following condition are equivalent.*

(1) $$\|T\|^2 + |\lambda|^2 \leq \|T - \lambda\|^2 \textit{for all} \lambda \in C^n$$

(2) $$\|T\|^2 \leq \|T + \lambda\| \textit{for all} \lambda \in C^n.$$

THEOREM 2 (TAKAGUCH [2]). *Given $T = (T_1, T_2, \ldots, T_n)$, there exists a unique $z_0 \in C^n$ such that*

$$\|T - z_0\| \leq \|T - \lambda\|$$

for all $\lambda \in C^n$.

THEOREM 3. *Let $T = (T_1, T_2, \ldots, T_n)$ be an n–tuple of operators. Then there exists a unique $z_0 \in C^n$ such that*

$$\|T - z_0\|^2 + |\lambda|^2 \leq \|(T - z_0) + \lambda\|^2$$

for all $\lambda \in C^n$.

Proof. By Theorem 1 and Theorem 2, there exists a $z_0 \in C^n$ such that

$$\|T - z_0\| \leq \|(T - z_0) + \lambda\|$$

for all $\lambda \in C^n$.

DEFINITION 1.

Given $T = (T_1, T_2, \ldots, T_n)$, we define $m_T = (m_{T_1}, m_{T_2}, \ldots, m_{T_n})$ the center of mass (center) of T to be the point z_0 specified in Theorem 3.

THEOREM 4 (TAKAGUCH [2]). *Let $T \in B(H)^n$ be a commuting jointly transloid, and let $\sigma_\bullet(T)$ be $\sigma_\pi(T), \sigma_l(T), \sigma_H(T), \sigma_T(T), \sigma'(T)$, $\sigma''(T)$, or $\sigma_P(T)$. Then m_T is the center of the smallest closed ball containing $\sigma_\bullet(T)$.*

3. JOINTLY CENTROID OPERATORS.

DEFINITION 2.

Let D_T be the smallest closed ball containing $\sigma(T)$, z_T its center, R_T its radius and ∂_T its boundary.

Let W_T be the radius of the smallest closed ball containing jointly numerical range and w_T its center.

THEOREM 5. *For an n-tuple of operators $T = (T_1, T_2, \ldots, T_n)$, if $0 \in W_0(T)$, then $\|T\|^2 + |\lambda|^2 \le \|T + \lambda\|^2$ for all $\lambda \in C^n$. Conversely if $\|T\| \le \|T + \lambda\|$ for all $\lambda \in C^n$, then $0 \in W_0(T)$.*

Proof. If $0 \in W_0(T)$, then there exists $x_n \in H, \|x_n\| = 1$ such that

$$\begin{aligned}
&\|(T+\lambda)x_n\|^2 = \sum_{i=1}^{n} \|(T_i + \lambda_i)x_n\|^2 \\
&= \sum_{i=1}^{n}((T_i + \lambda_i)^*(T_i + \lambda_i)x_n, x_n) \\
&= \sum_{i=1}^{n}((T_i^* T_i x_n, x_n) + (T_i^* \lambda_i x_x, x_n) + \lambda_i^* T_i x_n, x_n) + |\lambda_i|^2) \\
&= \sum_{i=1}^{n} \|Tx_n\|^2 + \sum_{i=1}^{n} 2Re\lambda_i(T_i x_n, x_n) + \sum_{i=1}^{n} |\lambda_i|^2 \to \|T\|^2 + |\lambda|^2.
\end{aligned}$$

Hence

$$\|T + \lambda\|^2 \ge \|T\|^2 + |\lambda|^2$$

for all $\lambda \in C^n$.

Conversely, let

$$\|T\| \le \|T + \lambda\|$$

for all $\lambda \in C^n$ and $0 \notin W_0(T)$. Since

$$\|T\| \le \|T + \lambda\|$$

for all $\lambda \in C^n$ and

$$\begin{aligned}
&W_0(T) \\
&= \{(T_1 x_n, x_n), (T_2 x_n, x_n), \ldots, (T_n x_n, x_n) \to \lambda \in C^n : \|x_n\| = 1\},
\end{aligned}$$

We may assume that $ReW_0(T) \ge \tau > 0$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n), \tau_i > 0$ for each i. Let

$$D = \{x \in H : \|x\| = 1, ReW_0(T) \le \frac{\tau}{2}\}$$

and let

$$\eta = \sup\{(\sum_{i=1}^{n} \|T_i x\|^2)^{\frac{1}{2}} : x \in D\}.$$

Then $\eta \leq \|T\|$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in C^n$, μ_i is a positive real number such that

$$|\mu| = \min\{\sum_{i=1}^{n}(\frac{\tau_i}{2}), \frac{(\|T\| - \eta)}{2}\},$$

where $|\mu| = (\sum_{i=1}^{n} |\mu_i|^2)^{\frac{1}{2}}$. Consider $T - \mu$. If $x \in D$,then

$$\|(T - \mu)x\| \leq \|Tx\| + |\mu| \leq \eta + \mu \leq \|T\|.$$

Let $T_i x = (\alpha_i + i\beta_i)x + y$, where $x \notin D, \|x\| = 1$ and $(x, y) = 0$. Then

$$\begin{aligned}
&\sum_{i=1}^{n} \|(T_i - \mu_i)x\|^2 \\
&= \sum_{i=1}^{n}(\alpha_i - \mu_i)^2 + \sum_{i=1}^{n} \beta_i^2 \sum_{i=1}^{n} \|y\|^2 \\
&= \sum_{i=1}^{n}((\alpha_i^2 + \beta_i^2 + \|y\|^2 + (\mu_i^2 - 2\alpha_i \mu_i)).
\end{aligned}$$

For $\alpha_i > \mu_i > 0$,

$$\sum_{i=1}^{n}(\|T_i x\|^2 + (\mu_i - 2\alpha_i \mu_i)) < \|T\|^2.$$

Hence

$$\sup\{\sum_{i=1}^{n} \|(T_i - \mu_i)x\|^2 : \|x\| = 1\} \leq \|T\|^2,$$

i.e.

$$\|T - \mu\|^2 \leq \|T\|^2.$$

DEFINITION 3.

T is said to be jointly centroid if $T - z_T$ is jointly normaloid.

REMARK.

Jointly transloid T is jointly cntroid. Let T be a jointly centroid and $S = T - z$. Then $S - z_S = (T - z) - (z_T - z) = T - z_T$. Thus S is also centroid. That is, every translate of jointly centroid operator is centroid.

THEOREM 6. *For any n–tuple of operators $T = (T_1, T_2, \ldots, T_n)$,*

$$B_T = \|T - m_T\|^2.$$

Proof. For any $z \in C^n$,

$$\begin{aligned}&\sum_{i=1}^{n}(\|T_i x\|^2 - |(T_i x, x)|^2)\\ &= \sum_{i=1}^{n}\{\|(T_i - z_i)x\|^2 - |((T_i - z_i)x, x)|^2\}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{i=1}^{n}\{\|T_i x\|^2 - |(T_i x, x)|^2\}\\ &\leq \sum_{i=1}^{n}\|(T_i - z_i)x\|^2.\end{aligned}$$

and

$$\begin{aligned}&\sup\{\sum_{i=1}^{n}\|T_i x\|^2 - |(T_i x, x)|^2 : \|x\| = 1\}\\ &\leq \sup\{\sum_{i=1}^{n}\|(T_i - m_{T_i})x\|^2 : \|x\| = 1\}.\end{aligned}$$

Thus,

$$B_T \leq \|T - m_T\|^2.$$

Conversely, since $0 \in W_0(T - m_T)$, there is a sequence $\{x_m\}$ such as $\|x_m\| = 1$,

$$((T_i - m_{T_i})x_m, x_m) \to 0$$

and

$$\|(T - m_T)x_m\| \to \|T - m_T\|.$$

Hence

$$\begin{aligned}
&\lim_{m\to 0}\{\sum_{i=1}^{n} \|(T_i - m_{T_i})x_m\|^2 - |((T_i - m_{T_i})x_m, x_m)|^2\} \\
&= \lim_{m\to 0}\{\sum_{i=1}^{n} \|(T_i - m_{T_i})x_m\|^2\} \\
&= \|T - m_T\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{\|x\|=1} \{\sum_{i=1}^{n} \|T_i x\|^2 - |(T_i x, x)|^2 : \|x\| = 1\} \\
&\geq \|T - m_T\|^2.
\end{aligned}$$

THEOREM 7. *The jointly transcendental radius of an n–tuple of operators T is equal to the distance between T and the scalars. This is,$M_T = \|T - m_T\|$.*

THEOREM 8. *A commuting n–tuple of operators $T = (T_1, T_2, \ldots, T_n)$ is jointly centroid if and only if $B_T = R_T^2$.*

LEMMA 1. *For a commuting n–tuple of operators $T = (T_1, T_2, \ldots, T_n)$, if $z \neq z_T$, then $\|T - z\| \neq R_T$.*

Proof. Assume $z_T = 0$. Then R_T is the jointly spectral radius $r(T)$ of T. Suppose $\|T - z\| = r(T) = R_T$ for some $z = 0$. $r(T - z) \leq \|T - z\| = r(T)$ which is impossible since D_T is the smallest closed ball containing $\sigma(T)$.

LEMMA 2. *Let T be a commuting n–tuple of operators. Then $\sigma(T) \cap \partial_T \neq \emptyset$.*

Proof. Suppose $\sigma(T) \cap \partial_T = \emptyset$. Since $\sigma(T)$ and ∂_T are compact subsets of C^n, let $d = dist(\sigma(T), \partial_T) > 0$. Then the closed ball with center z_T and radius $R_T - \frac{d}{2}$ contains $\sigma(T)$. This contradicts to the fact that d_T is the smallest closed ball containing $\sigma(T)$.

LEMMA 3. *Let T be a commuting n–tuple of operators. Then $r(T - z_T) = R_T$.*

Proof. By Lemma 1, $|z_T - \lambda| = R_T$ for some $\lambda \in \sigma(T)$. Since $|z_T - \lambda| \leq R_T$ for all $\lambda \in \sigma(T)$, R_T is the radius of the smallest closed ball containing the spectrum $\sigma(T)$ and by the definition of the spectral radius , $r(T - z_T) = R_T$.

Proof of Theorem 8. Since

$$\begin{aligned}
&\sum_{i=1}^{n}(\|T_i x\|^2 - |(T_i x, x)|^2) \\
&= \sum_{i=1}^{n}\{\|(T_i - z_i)x\|^2 - |((T_i - z_i)x, x)|^2\},
\end{aligned}$$

for any $z \in C^n$, if we take $z = z_T$ and sup,

$$\begin{aligned}
&\sup\{\sum_{i=1}^{n}(\|T_i x\|^2 - |(T_i x, x)|^2) : \|x\| = 1\} \\
&= \sup\{\sum_{i=1}^{n}(\|T_i - z_{T_i} x\|^2 - |((T_i - z_{T_i})x, x)|^2) : \|x\| = 1\} \\
&\leq \sup\{\sum_{i=1}^{n} \|(T_i - z_{T_i})x\|^2 : \|x\| = 1\} \\
&= \|T - z_T\|^2.
\end{aligned}$$

Since T is jointly centroid, $r(T - z_T)^2 = R_T^2$ by Lemma 3.

Thus $B_T \leq R_T^2$. Conversely, if $B_T = R_T^2$, then $\|T - m_T\|^2 = R_T^2$. Hence $\|T - m_T\| = R_T$ and $m_T = z_T$ by Lemma 1. Thus $\|T - z_T\| = R_T$ or $T - z_T$ is jointly normaloid. Hence T is jointly centroid.

REMARK.

In Theorem 8 and above Lemma, the condition that T is commuting is not necessary in respect of $\sigma_\pi(T), \sigma_l(T)$ and $\sigma_H(T)$.

EXAMLPE.

Let $T = (T_1, T_2)$ with

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

Then we can easily show that $\sigma(T) = \{(1,0),(1,1)\}$ and $z_T = (1, \frac{1}{2})$. But $r(T - z_T) \neq \|T - z_T\|$. Hence T is not jointly centroid.

COROLLARY. *For a jointly centroid operator T, the center of mass of T coincides with z_T.*

REMARK.

The class of jointly centroid operators contains the class of jointly transloid operators.

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