FUZZY IRREDUCIBLE IDEALS IN \( \Gamma \)-RINGS

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In 1965, Zadeh [11] introduced the notion of fuzzy sets in a set \( S \) as a function from \( S \) into \([0,1]\). Rosenfeld [9] applied this concept to the theory of groupoids and groups. In [6], Kumar discussed the fuzzy irreducible ideals in rings. Motivated by the study of Kumar, we discuss, in this paper, the fuzzy irreducible ideals in \( \Gamma \)-rings.

We first review some fuzzy logic concepts. For any fuzzy sets \( \mu \) and \( \nu \) in a set \( S \), we define

\[
\mu \subseteq \nu \iff \mu(x) \leq \nu(x) \quad \text{for all } x \in X, \\
(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \text{for all } x \in X.
\]

Let \( \mu \) be any fuzzy set in a set \( S \). The set

\[
\mu_t = \{x \in X : \mu(x) \geq t\}, \quad \text{where } t \in [0,1],
\]

is called a level subset of \( \mu \).

Let \( S \) and \( S' \) be any two sets and let \( f : S \to S' \) be any function. If \( \mu \) is any fuzzy set in \( S \), then the fuzzy set \( \nu \) in \( S' \) defined by

\[
\nu(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, y \in S', \\
0 & \text{otherwise},
\end{cases}
\]

is called the image of \( \mu \) under \( f \), denoted by \( f(\mu) \). If \( \nu \) is a fuzzy set in \( f(S) \), then the fuzzy set \( \mu \) in \( S \) defined by \( \mu(x) = \nu(f(x)) \) for all \( x \in S \) is called the preimage of \( \nu \) under \( f \) and is denoted by \( f^{-1}(\nu) \). A fuzzy set \( \mu \) in \( S \) is said to be \( f \)-invariant if

\[
f(x) = f(y) \implies \mu(x) = \mu(y), \quad \text{where } x, y \in S.
\]

A fuzzy set \( \mu \) in a set \( S \) has sup property if, for any subset \( T \) of \( S \), there exists \( x_0 \in T \) such that

\[
\mu(x_0) = \sup_{t \in T} \mu(t).
\]

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LEMMA 1 ([9]). Let \( f \) be a function defined on a set \( S \). Then
(a) \( \mu \subseteq f^{-1}(f(\mu)) \) for any fuzzy set \( \mu \) in \( S \),
(b) \( \mu = f^{-1}(f(\mu)) \) provided that \( \mu \) is \( f \)-invariant for any fuzzy set \( \mu \) in \( S \),
(c) \( \mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2) \) for any fuzzy sets \( \mu_1, \mu_2 \) in \( S \),
(d) \( f(f^{-1}(\nu)) = \nu \) for any fuzzy set \( \nu \) in \( f(S) \),
(e) \( \nu_1 \subseteq \nu_2 \Rightarrow f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2) \) for any fuzzy sets \( \nu_1, \nu_2 \) in \( f(S) \).

DEFINITION 1 ([1]). If \( M = \{x, y, z, \ldots \} \) and \( \Gamma = \{\alpha, \beta, \gamma, \ldots \} \) are additive abelian groups, and for all \( x, y, z \) in \( M \) and all \( \alpha, \beta \) in \( \Gamma \), the following conditions are satisfied
(1) \( x\alpha y \) is an element of \( M \),
(2) \( (x + y)\alpha z = x\alpha z + y\alpha z \), \( x(\alpha + \beta)y = x\alpha y + x\beta y \), \( x\alpha(y + z) = x\alpha y + x\alpha z \),
(3) \( (x\alpha y)\beta z = x\alpha(y\beta z) \),
then \( M \) is called a \( \Gamma \)-ring.

In what follows, \( M \) and \( M' \) would mean \( \Gamma \)-rings unless otherwise specified.

DEFINITION 2 ([1]). A subset \( A \) of \( M \) is a left (right) ideal of \( M \) if \( A \) is an additive subgroup of \( M \) and
\[
M \Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\} (A \Gamma M)
\]
is contained in \( A \). If \( A \) is both a left and a right ideal, then \( A \) is a two-sided ideal, or simply an ideal of \( M \).

DEFINITION 3 ([2]). An ideal \( P \) of \( M \) is said to be prime if for every ideals \( A, B \) of \( M \), \( A \Gamma B \subseteq P \) implies \( A \subseteq P \) or \( B \subseteq P \).

PROPOSITION 1 ([2]). Let \( P \) be an ideal of \( M \). Then the following are equivalent:
(a) \( P \) is a prime ideal of \( M \).
(b) For all \( x, y \in M, x \Gamma M \Gamma y \subseteq P \) implies \( x \in P \) or \( y \in P \).

DEFINITION 4 ([5]). A fuzzy set \( \mu \) in \( M \) is called a fuzzy left (right) ideal of \( M \) if
(4) \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \),
(5) \( \mu(x\alpha y) \geq \mu(y) \) (\( \mu(x\alpha y) \geq \mu(x) \)).
for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set $\mu$ in $M$ is called a fuzzy ideal of $M$ if $\mu$ is both a fuzzy left and a fuzzy right ideal of $M$.

We note that $\mu$ is a fuzzy ideal of $M$ if and only if

1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
2. $\mu(x \alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

**Lemma 2 ([5]).** A fuzzy set $\mu$ in $M$ is a fuzzy ideal of $M$ if and only if the level subsets $\mu_t$, $t \in \text{Im}(\mu)$, are ideals of $M$.

**Remark 1.** It follows from [5; Theorem 5] that the level ideals of a fuzzy ideal $\mu$ need not be distinct. Moreover, the level ideals form a chain. As $\mu(x) \leq \mu(0)$ for all $x \in M$, therefore the level ideal $\mu_t$, $t = \mu(0)$, is smallest in the family of all level ideals of $\mu$. If $\text{Im}(\mu) = \{t_0, t_1, ..., t_n\}$ with $t_0 > t_1 > ... > t_n$, then the chain of level ideals of $\mu$ is given by

$$\mu_{t_0} \subset \mu_{t_1} \subset ... \subset \mu_{t_n} = M.$$ 

**Definition 5 ([4]).** Let $\mu$ and $\nu$ be fuzzy sets in $M$ and let $\alpha \in \Gamma$. The product $\mu \Gamma \nu$ is defined by $\mu \Gamma \nu(x) = \sup_{z = y \alpha z} \{\min\{\mu(z), \nu(z)\}\}$ and $\mu \Gamma \nu(x) = 0$ if $x$ is not expressible as $x = y \alpha z$.

**Proposition 2 ([4]).** Let $\mu$ and $\nu$ be fuzzy left ideals of $M$. Then $\mu \cap \nu$ is a fuzzy left ideal of $M$ (similar results hold for fuzzy right ideals and fuzzy ideals). If $\mu$ is a fuzzy right ideal and $\nu$ a fuzzy left ideal, then $\mu \Gamma \nu \subseteq \mu \cap \nu$.

**Definition 6 ([4]).** A fuzzy ideal $\mu$ of $M$ is said to be prime if

1. $\mu$ is not a constant function,
2. for any fuzzy ideals $\nu, \rho$ in $M$, $\nu \Gamma \rho \subseteq \mu$ implies $\nu \subseteq \mu$ or $\rho \subseteq \mu$.

**Lemma 3 ([4]).** If $\mu$ is any nonempty fuzzy set in $M$, then $\mu$ is a fuzzy prime ideal of $M$ if and only if $\text{Im}(\mu) = \{t_0, t_1\}$ where $t_0 = 1$ and $t_1 \in [0, 1)$, and the ideal $\mu_{t_0} = \{x \in M | \mu(x) = t_0 = 1\}$ is prime.
DEFINITION 7. An ideal \( A \) of \( M \) is said to be irreducible if for any ideals \( I \) and \( J \) of \( M \),
\[
A = I \cap J \quad \text{implies} \quad I = A \quad \text{or} \quad J = A.
\]

DEFINITION 8. A fuzzy ideal \( \mu \) of \( M \) is said to be fuzzy irreducible if it is not an intersection of two fuzzy ideals of \( M \) properly containing \( \mu \).

THEOREM 1. If \( \mu \) is any fuzzy prime ideal of \( M \), then \( \mu \) is fuzzy irreducible.

Proof. Assume that \( \mu \) is not fuzzy irreducible. Then there exist fuzzy ideals \( \nu \) and \( \lambda \) of \( M \) such that \( \mu = \nu \cap \lambda \), \( \mu \subseteq \nu \) and \( \mu \subseteq \lambda \). From \( \mu \subseteq \nu \) and \( \mu \subseteq \lambda \), we have that \( \mu(x) < \nu(x) \) and \( \mu(y) < \lambda(y) \) for some \( x, y \in M \). If \( \mu \) is constant, then \( \mu(x) = \mu(y) = \mu(x\alpha y) \) for all \( \alpha \in \Gamma \). Since
\[
(\nu \cap \lambda)(x\alpha y) \geq \min\{\nu(x), \lambda(y)\}
\]
\[
> \min\{\mu(x), \mu(y)\}
\]
\[
= \mu(x\alpha y),
\]
it follows from Proposition 2 that \( (\nu \cap \lambda)(x\alpha y) > \mu(x\alpha y) \). This is a contradiction. If \( \mu \) is nonconstant, then by Lemma 3, \( \text{Im}(\mu) = \{t_0, t_1\} \) where \( t_0 = 1 \) and \( t_1 \in [0, 1) \); and the ideal \( \mu_{t_0} = \{x \in M | \mu(x) = t_0 = 1\} \) is prime. Following Remark 1, we have that the chain of level ideals of \( \mu \) is \( \mu_{t_0} \subseteq M \). Thus there exists \( s \in [0, 1) \) such that \( \mu(x) = s \) for all \( x \in M \setminus \mu_{t_0} \). Since \( \nu(x) > \mu(x) \) and \( \lambda(y) > \mu(y) \), therefore \( x, y \not\in \mu_{t_0} \). From the fact that \( \mu_{t_0} \) is a prime ideal of \( M \), it follows that \( x \Gamma M \setminus y \not\subseteq \mu_{t_0} \), i.e., \( x\alpha z \beta y \not\subseteq \mu_{t_0} \) for all \( z \in M \) and all \( \alpha, \beta \in \Gamma \), so that \( \mu(x\alpha z \beta y) = s \). On the other hand
\[
(\nu \cap \lambda)(x\alpha z \beta y) \geq \min\{\nu(x\alpha z), \lambda(y)\}
\]
\[
\geq \min\{\max\{\nu(x), \nu(z)\}, \lambda(y)\}
\]
\[
= \min\{\nu(x), \lambda(y)\}
\]
\[
> \min\{\mu(x), \mu(y)\}
\]
\[
= s = \mu(x\alpha z \beta y).
\]
Hence \( (\nu \cap \lambda)(x\alpha z \beta y) > \mu(x\alpha z \beta y) \), a contradiction. This completes the proof.
THEOREM 2. Let $\mu$ be any non-constant fuzzy irreducible ideal of $M$. Then

(a) $1 \in \text{Im}(\mu)$,
(b) there exists $t \in [0,1)$ such that $\mu(x) = t$ for all $x \in M - \mu_{t_0}$,
where $\mu_{t_0} = \{ x \in M | \mu(x) = 1 \}$, $t_0 = \mu(0) = 1$,
(c) the ideal $\mu_{t_0}$ is irreducible.

Proof. (a). Assume that $1 \notin \text{Im}(\mu)$. Then $\mu(0) < 1$, say $\mu(0) = t_0$. Define fuzzy sets $\mu_1$ and $\mu_2 : M \to [0,1]$ by

$$
\mu_1(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
\mu(x) & \text{otherwise}
\end{cases}
$$

and $\mu_2(x) = \mu(0)$ for all $x \in M$. It follows from Lemma 2 that $\mu_1$ and $\mu_2$ are fuzzy ideals of $M$. We now show that $\mu = \mu_1 \cap \mu_2$. If $x \in \mu_{t_0}$ then $\mu(x) \geq t_0 = \mu(0)$, and so $\mu(x) = \mu(0)$. But

$$
(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}
= \min\{1, \mu(0)\}
= \mu(0)
= \mu(x).
$$

If $x \in M - \mu_{t_0}$ then

$$
(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} = \min\{\mu(x), \mu(0)\} = \mu(x).
$$

Hence $\mu = \mu_1 \cap \mu_2$. Clearly $\mu \subset \mu_1$ and $\mu \subset \mu_2$. This contradicts the fact that $\mu$ is fuzzy irreducible. Therefore $1 \in \text{Im}(\mu)$, so that $t_0 = \mu(0) = 1$.

(b). It is sufficient to show that the chain of level ideals of $\mu$ is precisely $\mu_{t_0} \subset M$. Let $\mu_t, t \in [0,1)$, be any level ideal of $\mu$ such that $\mu_{t_0} \subset \mu_t \subset M$. Then there exists $s_1 \in (0, t)$ such that

$$
\mu(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
t & \text{if } x \in \mu_t - \mu_{t_0}, \\
s_1 & \text{if } x \in M - \mu_t.
\end{cases}
$$
Define fuzzy sets \( \mu_3 \) and \( \mu_4 : M \to [0, 1] \) as follows:

\[
\mu_3(x) = \begin{cases} 
1 & \text{if } x \in \mu_t, \\
 s_1 & \text{otherwise,} 
\end{cases} \quad \mu_4(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
t & \text{if } x \in \mu_t - \mu_{t_0}, \\
s_2 & \text{if } x \in M - \mu_t, 
\end{cases}
\]

where \( s_1 < s_2 < t \). By routine calculations, we know that \( \mu_3 \) and \( \mu_4 \) are fuzzy ideals of \( M \). Next we show that \( \mu = \mu_3 \cap \mu_4 \). If \( x \in \mu_{t_0} \) then

\[
(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = 1 = \mu(x).
\]

If \( x \in \mu_t - \mu_{t_0} \) then

\[
(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = t = \mu(x).
\]

If \( x \in M - \mu_t \) then

\[
(\mu_3 \cap \mu_4)(x) = \min\{\mu_3(x), \mu_4(x)\} = s_1 = \mu(x).
\]

Thus \( \mu = \mu_3 \cap \mu_4 \). It is clear that \( \mu \subset \mu_3 \) and \( \mu \subset \mu_4 \). This contradicts the fact that \( \mu \) is fuzzy irreducible. Hence the chain of level ideals of \( \mu \) is \( \mu_{t_0} \subset M \), so that

\[
\mu(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
s & \text{if } x \in M - \mu_{t_0}, 
\end{cases}
\]

for some \( s \in [0, 1) \).

(c). Assume that \( \mu_{t_0} \) is not irreducible. Then there exist ideals \( A \) and \( B \) of \( M \) such that \( \mu_{t_0} = A \cap B, \mu_{t_0} \subset A, \mu_{t_0} \subset B \). Thus \( A \) is not contained in \( B \) and \( B \) is not contained in \( A \), and \( (A - \mu_{t_0}) \cap (B - \mu_{t_0}) \) is the empty set. Let \( \mu_5 \) and \( \mu_6 \) be fuzzy sets in \( M \) defined by

\[
\mu_5(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
t_0' & \text{if } x \in A - \mu_{t_0}, \\
s & \text{if } x \in M - A, 
\end{cases} \quad \mu_6(x) = \begin{cases} 
1 & \text{if } x \in \mu_{t_0}, \\
t_0' & \text{if } x \in B - \mu_{t_0}, \\
s & \text{if } x \in M - B, 
\end{cases}
\]

where \( s < t_0' < 1 \). Then \( \mu_5 \) and \( \mu_6 \) are fuzzy ideals of \( M \) satisfying

\[
\mu = \mu_5 \cap \mu_6, \mu \subset \mu_5, \mu \subset \mu_6.
\]

This is impossible as \( \mu \) is fuzzy irreducible. The proof is complete.
Definition 9. A mapping $f : M \to M'$ is called a $\Gamma$-homomorphism if $f(x + y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)f(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Lemma 4 ([5]). (a) A $\Gamma$-homomorphic preimage of a fuzzy left (right) ideal is a fuzzy left(right) ideal. (b) A $\Gamma$-homomorphic image of a fuzzy left(right) ideal which has sup property is a fuzzy left(right) ideal.

Theorem 3. Let $f : M \to M'$ be a $\Gamma$-homomorphism. Then
(a) if $\mu$ is a $f$-invariant fuzzy irreducible ideal of $M$ with sup property, then $f(\mu)$ is a fuzzy irreducible ideal of $M'$.
(b) if $\mu'$ is any fuzzy irreducible ideal of $M'$ and if every fuzzy ideal of $M$ is $f$-invariant, then $f^{-1}(\mu')$ is a fuzzy irreducible ideal of $M$.

Proof. By Lemma 4, $f(\mu)$ and $f^{-1}(\mu)$ are fuzzy ideals of $M'$ and $M$ respectively. Assume that $f(\mu)$ is not fuzzy irreducible. Then there exist fuzzy ideals $\nu_1$ and $\nu_2$ of $M'$ such that $f(\mu) = \nu_1 \cap \nu_2, f(\mu) \subset \nu_1, f(\mu) \subset \nu_2$. As $\mu$ is $f$-invariant, it follows from Lemma 1 that

$$\mu = f^{-1}(\nu_1 \cap \nu_2), \mu \subset f^{-1}(\nu_1) \text{ and } \mu \subset f^{-1}(\nu_2).$$

To show that $f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2)$, let $x$ be any element of $M$. Then

$$(f^{-1}(\nu_1 \cap \nu_2))(x) = (\nu_1 \cap \nu_2)(f(x))$$

$$= \min\{\nu_1(f(x)), \nu_2(f(x))\}$$

$$= \min\{(f^{-1}(\nu_1))(x), (f^{-1}(\nu_2))(x)\}$$

$$= (f^{-1}(\nu_1) \cap f^{-1}(\nu_2))(x),$$

which implies that

$$f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2).$$

Hence $\mu = f^{-1}(\nu_1) \cap f^{-1}(\nu_2), \mu \subset f^{-1}(\nu_1), \mu \subset f^{-1}(\nu_2)$, which contradicts the fact that $\mu$ is fuzzy irreducible. Therefore $f(\mu)$ is fuzzy irreducible, which proves (a).
To prove (b), assume that $f^{-1}(\mu')$ is not fuzzy irreducible. Then $f^{-1}(\mu') = \sigma_1 \cap \sigma_2$, $f^{-1}(\mu') \subseteq \sigma_1$ and $f^{-1}(\mu') \subseteq \sigma_2$ of $M$. It is evident from Lemma 1 that $\mu' = f(\sigma_1 \cap \sigma_2)$, $\mu' \subseteq f(\sigma_1)$ and $\mu' \subseteq f(\sigma_2)$. Now we show that $f(\sigma_1 \cap \sigma_2) = f(\sigma_1) \cap f(\sigma_2)$. Since $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$ and $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, it follows from Lemma 1(c) that $f(\sigma_1 \cap \sigma_2) \subseteq f(\sigma_1) \cap f(\sigma_2)$. To establish the reverse inclusion, let $y \in M'$, $t = (f(\sigma_1) \cap f(\sigma_2))(y)$ and $\epsilon > 0$ be any real number. Then

$$t - \epsilon < \min\{ (f(\sigma_1))(y), (f(\sigma_2))(y) \}$$

$$= \min\{ \sup_{x \in f^{-1}(y)} \sigma_1(x), (f(\sigma_2))(y) \},$$

which implies that $t - \epsilon < \sigma_1(z)$ for some $z \in f^{-1}(y)$ and $t - \epsilon < (f(\sigma_2))(y)$. This means that $t - \epsilon < \sigma_1(z)$ and

$$t - \epsilon < (f(\sigma_2))(f(z))$$

$$= (f^{-1}(f(\sigma_2)))(z)$$

$$= \sigma_2(z) \text{ since } \sigma_2 \text{ is } f\text{-invariant}.$$

Hence

$t - \epsilon < \min\{ \sigma_1(z), \sigma_2(z) \} = (\sigma_1 \cap \sigma_2)(z).$

From $z \in f^{-1}(y)$ it follows that

$$t - \epsilon < \sup_{x \in f^{-1}(y)} (\sigma_1 \cap \sigma_2)(x) = (f(\sigma_1 \cap \sigma_2))(y).$$

As $\epsilon > 0$ was arbitrary, therefore

$t = (f(\sigma_1) \cap f(\sigma_2))(y) \leq (f(\sigma_1 \cap \sigma_2))(y),$

so that

$f(\sigma_1) \cap f(\sigma_2) \subseteq f(\sigma_1 \cap \sigma_2).$

Hence $\mu' = f(\sigma_1) \cap f(\sigma_2)$, $\mu' \subseteq f(\sigma_1)$ and $\mu' \subseteq f(\sigma_2)$, which contradicts the fact that $\mu'$ is fuzzy irreducible, The proof is complete.

The following theorem is an immediate consequence of Lemma 1 and Theorem 3.

**Theorem 4.** Let $f : M \rightarrow M'$ be a $\Gamma$-homomorphism and let every fuzzy ideal of $M$ be $f$-invariant. Then the mapping $\sigma \mapsto f(\sigma)$ defines a 1-1 correspondence between the set of all fuzzy irreducible ideals of $M$ with sup property and the set of all fuzzy irreducible ideals of $M'$. 
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