Pusan Kyŏngnam Math J. 9(1993), No. 2, pp. 295–311

DE RHAM COHOMOLOGY OF TOROIDAL GROUPS AND CHERN CLASSES OF THE COMPLEX LINE BUNDLES

TAKASHI UMENO

0. Introduction

Let $\mathbf{C}^n/\Gamma$ be a toroidal group of complex dimension n, where Γ is a discrete lattice of $\mathbf{C}^n$ generated by $\mathbf{R}$-lineary independent vectors $e_1, \cdots, e_n, v_1, \cdots, v_q$ over $\mathbf{Z}$ and e_i denotes the i-th unit vector of $\mathbf{C}^n$.

C.Vogt([6]) characterized the toroidal group $\mathbf{C}^n/\Gamma$ on which every complex line bundle is a theta bundle, investigating the theory of multipliers of complex line bundles on $\mathbf{C}^n/\Gamma$. In particular, he proved that the finite dimensionality of the cohomology group $H^1(\mathbf{C}^n/\Gamma, \mathcal{O})$ gives one of the characterizations.

On the other hand, we caluculated the $\bar{\partial}$ -cohomology groups of $\mathbf{C}^n/\Gamma$, using the Fourier expansions of (r,s) -forms on $\mathbf{C}^n/\Gamma$([2],[3]) In this paper, we shall apply these methods to the caluculation of de Rham cohomology of $\mathbf{C}^n/\Gamma$ and get several conditions for a $\mathbf{Z}$-valued skew-symmetric form E on Γ to be the Chern class of some complex line bundle on $\mathbf{C}^n/\Gamma$. Further, we shall show the existence of some special class of hermitian forms which define complex line bundles and prove that the hermitian form is uniquely determined by a complex line bundle on $\mathbf{C}^n/\Gamma$.

The author wishes to thank Professor Kazama for fruitful discussions during the preparation of this paper. The author is very grateful to Professor Kajiwara for his several suggestions and encouragement.

Received October 21, 1993.

Dedicated to Professor Joji Kajiwara on his 60th birthday

1. Preliminaries

Throughout this paper we assume that $\mathbf{C}^n/\Gamma$ is a toroidal group of complex dimension n, where Γ is a discrete lattice of $\mathbf{C}^n$ generated by $\mathbf{R}$-lineary independent vectors $\{e_1, \cdots, e_n, v_1 = {}^t(v_{11}, \cdots, v_{n1}), \cdots, v_q = {}^t(v_{1q}, \cdots, v_{nq})\}$ over $\mathbf{Z}$ and e_i denotes the i-th unit vector of $\mathbf{C}^n$. We may assume det $[\text{Im } v_{ij}; 1 \leq i, j \leq q] \neq 0$. We put $v_i = \sqrt{-1}e_i$ for $q + 1 \leq i \leq n$. Put

(1.1)
$$V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \cdots, v_q],$$
$$V_1 = [v_{ij}; 1 \leq i, j \leq q], \text{ and } V_2 = [v_{ij}; q + 1 \leq i \leq n, 1 \leq j \leq q].$$

We set $K_{m,i} := \sum_{j=1}^{n}(m_j v_{ji} - m_{n+i})$ and $K_m := \max\{|K_{m,i}|; 1 \leq i \leq q\}$ for $m = {}^t(m_1, \cdots, m_{n+q}) \in \mathbf{Z}^{n+q}$. Since $\mathbf{C}^n/\Gamma$ is toroidal, $K_m > 0$ for any $m \in \mathbf{Z}^{n+q}\backslash\{0\}$([4]).

Definition 1.1. We say that a toroidal group $\mathbf{C}^n/\Gamma$ is of finite type if $\mathbf{C}^n/\Gamma$ satisfies the following condition :

There exists $a > 0$ such that
$$\sup_{m \neq 0}\{\exp(-a\|m^*\|)/K_m\} < \infty, \quad \text{where } \|m^*\| = \max\{|m_i|; 1 \leq i \leq n\}.$$

By the results of [3], a toroidal group $\mathbf{C}^n/\Gamma$ of finite type satisfies for $1 \leq r \leq n$,

$$\dim H^s(\mathbf{C}^n/\Gamma, \Omega^r) = \begin{cases} \binom{n}{r}\binom{q}{s}, & \text{if } 1 \leq s \leq q \\ 0, & \text{if } s > q. \end{cases} \tag{1.2}$$

We put $\beta_i = \text{Im}v_i$ for $1 \leq i \leq n$ and $\beta = [\beta_{ij}] := [\beta_1, \cdots \beta_n]$. Then $\beta_1, \cdots \beta_n$ are lineary independent over $\mathbf{C}$ and we put $\gamma = [\gamma_{ij}] := \beta^{-1}$. For any $z \in \mathbf{C}^n$, we define two coordinates $z_1, \cdots, z_n$ and $t_1, \cdots, t_{2n}$ by

$$\begin{aligned} z &= z_1\beta_1 + \cdots + z_n\beta_n \\ &= t_1e_1 + \cdots + t_ne_n + t_{n+1}v_1 + \cdots + t_{2n}v_n. \end{aligned} \tag{1.3}$$

Then we have for $i = 1, \cdots, n$,

$$(1.4) \qquad t_i = \frac{1}{2\sqrt{-1}}\left(-\sum_{j=1}^{n} \overline{v}_{ij} z_j + \sum_{j=1}^{n} v_{ij} \overline{z}_j\right) \quad \text{and}$$

$$t_{n+i} = \frac{1}{2\sqrt{-1}}(z_i - \overline{z}_i).$$

These coordinates $z = {}^t(z_1, \cdots, z_n)$ and $t = {}^t(t_1, \cdots, t_{2n})$ define local coordinates in $\mathbf{C}^n/\Gamma$. The mapping $\phi : \mathbf{C}^n \ni z = {}^t(z_1, \cdots, z_n) \mapsto t = {}^t(t_1, \cdots, t_{2n}) \in \mathbf{R}^{2n}$ induces an isomorphism as a real Lie group $\phi : \mathbf{C}^n/\Gamma \mapsto \mathbf{R}^{2n}/\phi(\Gamma) = \mathbf{T}^{n+q} \times \mathbf{R}^{n-q}$, where $\mathbf{T}^{n+q}$ is a real torus of real dimension $n+q$. For $t = {}^t(t_1, \cdots, t_{2n}) \in \mathbf{R}^{2n}$ and $m = {}^t(m_1, \cdots, m_{n+q}) \in \mathbf{Z}^{n+q}$, we put $t' = {}^t(t_1, \cdots, t_{n+q}), t'' = {}^t(t_{n+q+1}, \cdots, t_{2n})$ and $< m, t' > := m_1 t_1 + \cdots + m_{n+q} t_{n+q}$. Let f be a complex valued $\mathcal{C}^\infty$ function on $\mathbf{C}^n/\Gamma$. Then we have the Fourier expansion of f :

(1.5)

$$f(t) = \sum_{m \in \mathbf{Z}^{n+q}} a^m(t'') \exp 2\pi\sqrt{-1} < m, t' > \quad \text{for} \quad t = \begin{pmatrix} t' \\ t'' \end{pmatrix} \in \mathbf{R}^{2n}.$$

By the standard argument of Fourier analysis, a series $\sum_{m \in \mathbf{Z}^{n+q}} a^m(t'') \exp 2\pi\sqrt{-1} < m, t' >$ converges to a $\mathcal{C}^\infty$ function on $\mathbf{C}^n/\Gamma$ if and only if

$$(1.6) \qquad C(\ell, I, R) := \sup_{|t''| \le R} \left\{ \left| \frac{\partial^\ell a^m(t'')}{\partial t''^\ell} \right| \|m\|^I ; m \in \mathbf{Z}^{n+q} \right\} < \infty,$$

for any positive integers ℓ, I and any positive number R, where $|t''| = \sqrt{t_{n+q+1}^2 + \cdots + t_{2n}^2}$ and $\|m\| = \max\{|m_i|; i = 1, \cdots, n+q\}$.

Let $T' := \mathbf{C}\left\{\frac{\partial}{\partial z_i}; i = 1, \cdots, n\right\}$ be the holomorphic tangent space of $\mathbf{C}^n/\Gamma$ at 0,

$T_{\mathbf{R}} := \mathbf{R}\left\{\frac{\partial}{\partial t_i}; i = 1, \cdots, 2n\right\}$ the real tangent space of $\mathbf{C}^n/\Gamma$ at 0,

$T_\Gamma := \mathbf{R}\left\{\frac{\partial}{\partial t_i}; i = 1, \cdots, n+q\right\}$, and $\mathbf{R}_\Gamma := \mathbf{R}\{e_1, \cdots, e_n, v_1, \cdots, v_q\}$.

For $\sigma = \sum_{i=1}^{n} \sigma_i \frac{\partial}{\partial z_i} \in T'$, we put $\widehat{\sigma} := \sigma + \overline{\sigma} = \sum_{i=1}^{2n} s_i \frac{\partial}{\partial t_i} \in T_{\mathbf{R}}$. Then the mapping

$$(1.7) \qquad T' \ni \sigma \longmapsto \widehat{\sigma} \in T_{\mathbf{R}} \quad \text{is an } \mathbf{R}\text{-isomorhism.}$$

From (1.3) and (1.4), we have

$$(1.8) \quad \sigma_1\beta_1 + \cdots + \sigma_n\beta_n = s_1e_1 + \cdots + s_ne_n + s_{n+1}v_1 + \cdots + s_{2n}v_n.$$

By the mappings

$$(1.9) \qquad \begin{aligned} T' \ni \sigma &\longmapsto {}^t(\sigma_1, \cdots, \sigma_n) \in \mathbf{C}^n \quad \text{and} \\ T_{\mathbf{R}} \ni \widehat{\sigma} &\longmapsto {}^t(s_1, \cdots, s_{2n}) \in \mathbf{R}^{2n}, \end{aligned}$$

we can identify T' with $\mathbf{C}^n$, $T_{\mathbf{R}}$ with $\mathbf{R}^{2n}$ and T_Γ with $\mathbf{R}_\Gamma$, respectively.

2. de Rham cohomology of toroidal groups

In this section, we calculate the de Rham cohomology groups of toroidal groups $\mathbf{C}^n/\Gamma$. Let $\mathcal{C}$ be the sheaf of germs of complex valued $\mathcal{C}^\infty$ functions on $\mathbf{C}^n/\Gamma$, $\mathcal{C}^p$ the sheaf of germs of $\mathcal{C}^\infty p$-forms on $\mathbf{C}^n/\Gamma$, and Ω^r the sheaf of germs of holomorphic r-forms on $\mathbf{C}^n/\Gamma$. We denote by $Z_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)$ the space of d-closed $\mathcal{C}^\infty p$-forms on $\mathbf{C}^n/\Gamma$ and by $B_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)$ the space of d-exact $\mathcal{C}^\infty p$-forms on $\mathbf{C}^n/\Gamma$. We have

$$H^p(\mathbf{C}^n/\Gamma, \mathbf{C}) = \frac{Z_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)}{B_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)}.$$

Let φ be a $\mathcal{C}^\infty p$-form on $\mathbf{C}^n/\Gamma$, we write

$$\varphi(t) = \frac{1}{p!} \sum_{1 \le i_1, \cdots, i_p \le 2n} \varphi_{i_1 \cdots i_p}(t) dt_{i_1} \wedge \cdots \wedge dt_{i_p}.$$

We expand $\varphi_{i_1 \cdots i_p}(t)$ as in (1.5) and put

$$\varphi_{i_1 \cdots i_p}(t) = \sum_{m \in \mathbf{Z}^{n+q}} a^m_{i_1 \cdots i_p}(t'') \exp 2\pi\sqrt{-1} < m, t' > \text{ and}$$
$$\varphi^m := \frac{1}{p!} \sum_{1 \le i_1, \cdots, i_p \le 2n} a^m_{i_1 \cdots i_p}(t'') \exp 2\pi\sqrt{-1} < m, t' > dt_{i_1} \wedge \cdots \wedge dt_{i_p}.$$

Then $\varphi = \sum_{m \in \mathbf{Z}^{n+q}} \varphi^m$. Suppose $\varphi \in B_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)$. There exists a $\mathcal{C}^\infty(p-1)$-form $\psi = \sum_{m \in \mathbf{Z}^{n+q}} \psi^m$ such that $\varphi = \overline{\partial}\psi$. Then we have $\varphi^m = \overline{\partial}\psi^m$ for any $m \in \mathbf{Z}^{n+q}$. We put

$$\psi^m = \frac{1}{(p-1)!} \sum_{1 \le i_1, \cdots, i_{p-1} \le 2n} b^m_{i_1 \cdots i_{p-1}}(t'') \exp 2\pi\sqrt{-1} < m, t' > dt_{i_1} \wedge \cdots \wedge dt_{i_{p-1}}.$$

The equation $\varphi = \overline{\partial}\psi$ implies for any $m \in \mathbf{Z}^{n+q}$ and $1 \le i_1 < \cdots < i_p \le 2n$,

$$a^m_{i_1 \cdots i_p}(t'') = \sum_{k=1}^{\ell} (-1)^{k+1} 2\pi\sqrt{-1} m_k b^m_{i_1 \cdots \hat{i}_k \cdots i_p}(t'') + \sum_{k=\ell+1}^{p} (-1)^{k+1} \frac{\partial b^m_{i_1 \cdots \hat{i}_k \cdots i_p}(t'')}{\partial t_{i_k}}, \tag{2.1}$$

where $\ell := \max\{k; i_k \le n+q\}$. In particular, we have

$$1 \le i_1, \cdots, i_p \le n+q \Rightarrow a^0_{i_1 \cdots i_p}(t'') \equiv 0. \tag{2.2}$$

Now suppose $\varphi \in Z_d(\mathbf{C}^n/\Gamma, \mathcal{C}^p)$. For each $m = (m_1, \cdots, m_{n+q}) \in \mathbf{Z}^{n+q} \backslash \{0\}$ we put $i(m) := \max\{i; m_i \neq 0\}$ and $M(m) := m_{i(m)}$. For any $1 \le i_1, \cdots, i_p \le 2n$ and $m \in \mathbf{Z} \backslash \{0\}$, we have

$$(2.3)\qquad 2\pi\sqrt{-1}M(m)a^m_{i_1\cdots i_p}(t'')\exp 2\pi\sqrt{-1} < m,t' >$$

$$= \sum_{k=1}^{p}(-1)^{k+1}\frac{\partial(a^m_{i(m)i_1\cdots\hat{i}_k\cdots i_p}(t'')\exp 2\pi\sqrt{-1} < m,t' >)}{\partial t_{i_k}}$$

We put

$$(2.4)\qquad b^m_{i_1\cdots i_{p-1}}(t'') := \frac{a^m_{i(m)i_1\cdots i_{p-1}}(t'')}{2\pi\sqrt{-1}M(m)} \quad \text{and}$$

$$\psi^m := \frac{1}{(p-1)!}\sum_{1\leq i_1,\cdots\leq 2n} b^m_{i_1\cdots i_{p-1}}(t'')\exp 2\pi\sqrt{-1} < m, t' > dt_{i_1}\wedge\cdots\wedge dt_{i_p}.$$

From (2.3) and (2.4), we have $\varphi^m = d\psi^m$, for any $m \in \mathbf{Z}^{n+q}\backslash\{0\}$. Further, from (1.6) and (2.4), $\widetilde{\psi} := \sum_{m\in\mathbf{Z}^{n+q}\backslash\{0\}} \psi^m$ converges in $H^0(\mathbf{C}^n/\Gamma, C^p)$. Hence we have the following

Lemma 2.1. *Let* $\varphi = \sum_{m\in\mathbf{Z}^{n+q}} \varphi^m$ *be a* $C^\infty d$*-closed p-form on* $\mathbf{C}^n/\Gamma$. *Then we have a* $C^\infty(p-1)$*-form* $\widetilde{\psi} = \sum_{m\in\mathbf{Z}^{n+q}\backslash\{0\}} \psi^m$ *defined by* (2.4) *satisfying* $\varphi = \varphi^0 + d\widetilde{\psi}$.

In case $m = 0$ we get the following

Lemma 2.2. *Let* $\varphi^0 = \frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq 2n} a^0_{i_1\cdots i_p}(t'')dt_{i_1}\wedge\cdots\wedge dt_{i_p}$ *be a* $C^\infty d$*-closed p-form on* $\mathbf{C}^n/\Gamma$. *Then there exists a unique p-form with constant coefficients* $\chi = \frac{1}{p!}\sum_{1\leq i_1,\cdots,i_p\leq n+q} c_{i_1\cdots i_p}dt_{i_1}\wedge\cdots\wedge dt_{i_p}$

and a $(p-1)$*-form* $\psi^0 = \dfrac{1}{(p-1)!} \sum_{1 \le i_1, \cdots, i_{p-1} \le 2n} b^0_{i_1 \cdots i_p}(t'') dt_{i_1} \wedge \cdots \wedge dt_{i_p}$ *satisfying* $\varphi^0 = \chi + d\psi^0$.

Proof. The uniqueness of χ immediately follows by (2.2). We shall show the existence of χ and ψ^0.

For each $1 \le i_1 < \cdots < i_{p+1} \le 2n$, we put $\ell := \max\{k; i_k \le n+q\}$. We have

$$\sum_{k=\ell+1}^{p+1} (-1)^{k+1} \frac{\partial a^0_{i_1 \cdots i_\ell i_{\ell+1} \cdots \hat{i}_k \cdots i_{p+1}}(t'')}{\partial t_{i_k}} = 0$$

In case $\ell = p$, for each $1 \le i_1 < \cdots < i_p \le n+q$ and $n+q \le i \le 2n$, we have $\dfrac{\partial a^0_{i_1 \cdots i_p}(t'')}{\partial t_{i_k}} = 0$. Hence $c_{i_1 \cdots i_p} := a^0_{i_1 \cdots i_p}(t'')$ are constant. Put

$$\chi := \frac{1}{p!} \sum_{1 \le i_1, \cdots, i_p \le n+q} c_{i_1 \cdots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}.$$

In case $\ell < p$, for each $1 \le i_1 < \cdots < i_\ell \le n+q$,

$$\varphi^0_{i_1 \cdots i_\ell} := \sum_{n+q+1 \le i_{\ell+1} < \cdots < i_p \le 2n} a^0_{i_1 \cdots i_\ell i_{\ell+1} \cdots i_p}(t'') dt_{i_{\ell+1}} \wedge \cdots \wedge dt_{i_p}$$

is d-closed p'-form in $\mathbf{R}^{n-q}$, where $p' = p - \ell$. Then we have $(p'-1)$-form on $\mathbf{R}^{n-q}$

$$\psi^0_{i_1 \cdots i_\ell} := \sum_{n+q+1 \le i_{\ell+1} < \cdots < i_{p-1} \le 2n} b^0_{i_1 \cdots i_\ell i_{\ell+1} \cdots i_{p-1}}(t'') dt_{i_{\ell+1}} \wedge \cdots \wedge dt_{i_{p-1}}$$

satisfying $d\psi^0_{i_1 \cdots i_\ell} = \varphi^0_{i_1 \cdots i_\ell}$. Put

$$\psi^0 := \sum_{\ell=0}^{p-1} (-1)^\ell \sum_{\substack{1 \le i_1 < \cdots < i_\ell \le n+q \\ n+q+1 \le i_{\ell+1} < \cdots < i_{p-1} \le 2n}} b^0_{i_1 \cdots i_\ell i_{\ell+1} \cdots i_{p-1}}(t'') \, dt_{i_1} \wedge \cdots \wedge dt_{i_{p-1}}.$$

We have $d\psi^0 = \varphi^0 - \chi$ Q·E·D

Summarizing lemma 2.1 and lemma 2.2, we have the following

Proposition 2.1. *Let φ be a $C^\infty d$-closed p-form on a toroidal group $\mathbf{C}^n/\Gamma$. Then there exists a unique p-form with constant coefficients*

$$\chi = \frac{1}{p!}\sum_{1\le i_1,\cdots,i_p\le n+q} c_{i_1\cdots i_p} dt_{i_1}\wedge\cdots\wedge dt_{i_p} \tag{2.5}$$

and a $C^\infty(p-1)$-form ψ on $\mathbf{C}^n/\Gamma$ satisfying $\varphi = \chi + d\psi$.

Notation For any $C^\infty d$-closed p-forms φ_1 and φ_2 on $\mathbf{C}^n/\Gamma$ we write $\varphi_1 \sim \varphi_2$ when φ_1 and φ_2 are cohomologuous, namely, there is a $C^\infty(p-1)$-form ψ on $\mathbf{C}^n/\Gamma$ such that $\varphi_1 - \varphi_2 = d\psi$.

Since t_{n+i} are global functions for $i = q+1,\cdots,2n$ and from (1.4), we have

(2.6)

$$dt_i \sim \frac{1}{2\sqrt{-1}}\left(-\sum_{j=1}^{q}\overline{v}_{ij}dz_j + \sum_{j=1}^{q} v_{ij}d\overline{z}_j\right) \text{ for } i = 1,\cdots,q,$$

$$dt_i \sim \frac{1}{2\sqrt{-1}}\left(-\sum_{j=1}^{q}\overline{v}_{ij}dz_j + 2\sqrt{-1}dz_i + \sum_{j=1}^{q} v_{ij}d\overline{z}_j\right)$$

$$\text{for } i = q+1,\cdots,n,$$

$$dt_{n+i} \sim \frac{1}{2\sqrt{-1}}(dz_i - d\overline{z}_i) \text{ for } i = 1,\cdots,q, \text{ and}$$

(2.7)

$$dz_i \sim d\overline{z}_i \text{ for } i = q+1,\cdots,n.$$

Conversely it is easy to show that $dz_i, d\overline{z}_j$, for $i = 1,\cdots,n$ and $j = 1,\cdots,q$ are cohomologuous to linear combinations of $dt_1,\cdots,dt_{n+q}$. Substituting (2.6) to (2.5), we get

(2.8)

$$\chi \sim \chi_{\mathbf{C}} := \sum_{r+s=p}\frac{1}{r!s!}\sum_{\substack{1\le i_1,\cdots,i_r\le n\\ 1\le j_1,\cdots,j_s\le q}} c'_{i_1\cdots i_r j_1,\cdots j_s} dz_{i_1}\wedge\cdots\wedge dz_{i_r}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_s},$$

where $\chi_{\mathbf{C}}$ is a constant p-form on $\mathbf{C}^n/\Gamma$. From (2.6), the mapping

$$\bigwedge^p \mathbf{C}\{dt_1,\cdots,dt_{n+q}\} \ni \chi \mapsto \chi_{\mathbf{C}} \in \bigwedge^p \mathbf{C}\{dz_1,\cdots,dz_n,d\overline{z}_1,\cdots,d\overline{z}_q\} \tag{2.9}$$

is one to one correspondence. Hence we get the following

Theorem 2.1. *Let $\mathbf{C}^n/\Gamma$ be a toroidal group where Γ is generated by $\{e_1, \cdot\cdot, e_n, v_1, \cdot\cdot, v_q\}$. Then we have :*

(1) *any cohomology class of φ in $Z_d(\mathbf{C}^n/\Gamma, C^p)$ is represented by constant forms $\chi \in \bigwedge^p \mathbf{C}\{dt_1, \cdots, dt_{n+q}\}$ with respect to the basis $\{dt_1, \cdot\cdot\cdot, dt_{n+q}\}$ and $\chi_{\mathbf{C}} \in \mathbf{C}\{dz_1, \cdot\cdot, dz_n, d\overline{z}_1, \cdot\cdot, d\overline{z}_q\}$ with respect to the basis $\{dz_1, \cdot\cdot, dz_n, d\overline{z}_1, \cdot\cdot, d\overline{z}_q\}$. Further these forms χ and $\chi_{\mathbf{C}}$ are uniquely determined by φ.*

$$
\begin{aligned}
(2) \qquad H^p(\mathbf{C}^n/\Gamma, \mathbf{C}) &\cong \bigwedge^p \mathbf{C}\{dt_1, \cdots, dt_{n+q}\} \\
&\cong \bigwedge^p \mathbf{C}\{dz_1, \cdot\cdot, dz_n, d\overline{z}_1, \cdot\cdot, d\overline{z}_q\} \\
&\qquad \text{for } 1 \le p \le n+q \\
&= 0 \ \text{ for } p \ge n+q+1.
\end{aligned}
$$

In (2.8), *we put*

$$
\chi^{r,s} := \frac{1}{r!s!} \sum_{\substack{1 \le i_1, \cdots, i_r \le n \\ 1 \le j_1, \cdots, j_s \le q}} c'_{i_1 \cdots i_r j_1, \cdots j_s} dz_{i_1} \wedge \cdot\cdot \wedge dz_{i_r} \wedge d\overline{z}_{j_1} \wedge \cdot\cdot \wedge d\overline{z}_{j_s},
$$

for $0 \le r \le n$, and $0 \le s \le q$. Since $\chi^{r,s}$ is $\overline{\partial}$-closed and from theorem 2.1 we get homomorphisms

$$
(2.10) \qquad \Phi^{r,s} : H^p(\mathbf{C}^n/\Gamma, \mathbf{C}) \ni [\chi] \longmapsto [\chi^{r,s}] \in H^s(\mathbf{C}^n/\Gamma, \Omega^r)
$$

for $0 \le r \le n$ and $0 \le s \le q$ such that $r+s=p$. In case $\mathbf{C}^n/\Gamma$ is a toroidal group of finite type, by[3]

$$
\begin{aligned}
H^s(\mathbf{C}^n/\Gamma, \Omega^r) \cong \\
&\mathbf{C}\{dz_{i_1} \wedge \cdot\cdot \wedge dz_{i_r} \wedge d\overline{z}_{j_1} \wedge \cdot\cdot \wedge d\overline{z}_{j_s}; \\
&1 \le i_1 < \cdot\cdot < i_r \le n \ , \ 1 \le j_1 < \cdot\cdot < j_s \le q\}
\end{aligned}
$$

Thus we have the following

Theorem 2.2 *Let $\mathbf{C}^n/\Gamma$ be a toroidal group of finite type where Γ is generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_q\}$. Then the homomorphisms $\Phi^{r,s}$ defined by (2.10) are onto for $0 \leq r \leq n$ and $0 \leq s \leq q$ such that $r+s=p$. Further we get a Hodge decomposition*

$$H^p(\mathbf{C}^n/\Gamma, \mathbf{C}) \cong \bigoplus_{\substack{r+s=p \\ 0\leq r\leq n, 0\leq s\leq q}} H^s(\mathbf{C}^n/\Gamma, \Omega^r)$$

We note C.Vogt([7]) also showed the Hodge decomposition of theorem 2.2 by comparing the complex dimensions of the above cohomology spaces.

3. Chern classes of complex line bundles over toroidal groups

In this section we shall study the condition for $E \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$ to be the Chern class of some complex line bundle L on $\mathbf{C}^n/\Gamma$, and describe L by E.

We put $\Gamma = \mathbf{Z}\{e_1, \cdots, e_n, v_1, \cdots, v_q\} = \mathbf{Z}\{u_1, \cdots, u_{n+q}\}$. We denote by
$\widehat{u}_i \in H_1(\mathbf{C}^n/\Gamma, \mathbf{Z})$ the loop with base point $[0] \in \mathbf{C}^n/\Gamma$ lifts to a path in $\mathbf{C}^n$ starting at 0 and ending at a point u_i, for each i. Since $\int_{\widehat{\mathbf{u}}_i} dt_j = \delta_{ij}$ for $1 \leq i, j \leq n+q$, we have

$$H^1(\mathbf{C}^n/\Gamma, \mathbf{Z}) \cong \mathbf{Z}\{dt_1, \cdots, dt_{n+q}\}, \quad \text{and} \tag{3.1}$$

$$H^p(\mathbf{C}^n/\Gamma, \mathbf{Z}) \cong \bigwedge^p \mathbf{Z}\{dt_1, \cdots, dt_{n+q}\}. \tag{3.2}$$

Let $E \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$, then we can write

$$E = \frac{1}{2} \sum_{1\leq i,j\leq n+q} E_{ij} dt_i \wedge dt_j, \tag{3.3}$$

where $[E_{ij}]$ is a $\mathbf{Z}$-valued skew-symmetric matrix. We denote by $\mathbf{Z}^{n\times n}$ (resp. $\mathbf{C}^{n\times n}$) the set of $n \times n$, $\mathbf{Z}$(resp. $\mathbf{C}$)-valued matrices. We put

$$[E_{ij}] = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} F_1 & F_2 \\ -{}^tF_2 & F_3 \end{bmatrix}, \text{ and} \tag{3.4}$$
$$E_2 = \begin{bmatrix} F_4 \\ F_5 \end{bmatrix},$$

where $E_1 \in \mathbf{Z}^{n\times n}, E_3, F_1 \in \mathbf{Z}^{q\times q}, F_3 \in \mathbf{Z}^{(n-q)\times(n-q)}$, *and* $F_4 \in \mathbf{Z}^{q\times(n-q)}$. *From theorem 2.1, we have a unique constant 2-form*
$E_{\mathbf{C}} \in \bigwedge^2 \mathbf{C}\{dz_1, \cdots, dz_n, d\overline{z}_1, \cdots, d\overline{z}_q\}$ *such that* $E \sim E_{\mathbf{C}}$ *in* $Z_d(\mathbf{C}^n/\Gamma, \mathcal{C}^2)$. *Substituting* (2.6) *to* (3.3), *we get*

$$(3.5) \qquad E_{\mathbf{C}} = E^{2,0} + E^{1,1} + E^{0,2}, \text{ where } E^{r,s} = \Phi^{r,s}(E).$$

We put

$$(3.6) \quad E^{2,0} = \frac{1}{2} \sum_{1\le i,j\le n} A_{ij} dz_i \wedge dz_j, \qquad E^{1,1} = \sum_{\substack{1\le i\le n \\ 1\le j\le q}} B_{ij} dz_i \wedge d\overline{z}_j \quad \text{and}$$

$$E^{0,2} = \frac{1}{2} \sum_{1\le i,j\le q} C_{ij} d\overline{z}_i \wedge d\overline{z}_j.$$

Put

$$(3.7) \qquad A = [A_{ij}] = \begin{bmatrix} A_1 & A_2 \\ -{}^tA_2 & A^3 \end{bmatrix}, \qquad B = [B_{ij}] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ and } C = [C_{ij}],$$

where $A_1, B_1 \in \mathbf{C}^{q\times q}, A_3 \in \mathbf{C}^{(n-q)\times(n-q)}$. *Then we have*

$$(3.8) \qquad \begin{aligned} A_1 &= -\frac{1}{4}({}^t\overline{V}E_1\overline{V} + {}^tE_2\overline{V} - {}^t\overline{V}E_2 + E_3), \\ A_2 &= \frac{\sqrt{-1}}{2}({}^t\overline{V_1}F_2 + {}^t\overline{V_2}F_3 + {}^tF_5), \\ A_3 &= F_3, \ B_1 = \frac{1}{4}({}^t\overline{V}E_1V + {}^tE_2V - {}^t\overline{V}E_2 + E_3), \\ B_2 &= \frac{\sqrt{-1}}{2}({}^tF_2V_1 - F_3V_2 + F_5) = -{}^t\overline{A_2}, \\ C &= -\frac{1}{4}({}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3) = \overline{A}_1. \end{aligned}$$

By the exact sequence

$$H^1(\mathbf{C}^n/\Gamma, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathbf{C}^n/\Gamma, \mathbf{Z}) \xrightarrow{\iota} H^2(\mathbf{C}^n/\Gamma, \mathcal{O}),$$

for any $E \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$, *there exists a line bundle* $L \in H^1(\mathbf{C}^n/\Gamma, \mathcal{O}^*)$ *such that* $c_1(L) = E$ *if and only if*

(3.9) $\quad \iota(E) = 0$ *in* $H^2(\mathbf{C}^n/\Gamma, \mathcal{O})$.

From (2.10), (3.9) *is equivalent to*

(3.10) $\quad E^{0,2}$ *is* $\overline{\partial}$*-exact.*

Since $E^{0,2}$ *is a constant form, from the lemma 2.1 of [2],* (3.10) *is equivalent to*

$$(3.11) \qquad {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0$$

From (3.6), (3.8) *and* (3.11), *we obtain*

$$(3.12) \qquad E_{\mathbf{C}} = \sum_{\substack{1\le i\le q\\ q+1\le j\le n}} A_{ij}dz_i \wedge dz_j + \frac{1}{2}\sum_{q+1\le i,j\le n} E_{ij}dz_i \wedge d\overline{z}_j + \sum_{1\le i,j\le q} B_{ij}dz_i \wedge d\overline{z}_j - \sum_{\substack{q+1\le i\le n\\ 1\le j\le q}} \overline{A}_{ji}dz_i \wedge d\overline{z}_j,$$

Further,

(3.13) $\quad B_1 = [B_{ij}; 1 \le i,j \le q]$ *is a skew-Hermitian matrix and*
$\quad F_3 = [E_{ij}; q+1 \le i,j \le n]$ *is a real skew-symmetric matrix.*

From (2.7), $E_{\mathbf{C}}$ *is cohomologuous to*

$$(3.14) \qquad E_{\mathbf{R}} := \sum_{1\le i,j\le q} B_{ij}dz_i \wedge d\overline{z}_j + \sum_{\substack{1\le i\le q\\ q+1\le j\le n}} A_{ij}dz_i \wedge d\overline{z}_j - \sum_{\substack{q+1\le i\le n\\ 1\le j\le q}} \overline{A}_{ji}dz_i \wedge d\overline{z}_j + \frac{1}{2}\sum_{q+1\le i,j\le n} E_{ij}dz_i \wedge d\overline{z}_j.$$

From (3.13), *we see that* $E_{\mathbf{R}}$ *is a real* $(1,1)$*-form with constant coefficients on* $\mathbf{C}^n/\Gamma$. *We set*

$$\mathcal{E}^{1,1} :=\{F; F = \sum_{1\leq i,j\leq q} F^1_{ij} dz_i \wedge d\overline{z}_j + \sum_{\substack{1\leq i\leq q\\ q+1\leq j\leq n}} F^2_{ij} dz_i \wedge d\overline{z}_j$$

$$- \sum_{\substack{q+1\leq i\leq n\\ 1\leq j\leq q}} \overline{F}^2_{ji} dz_i \wedge d\overline{z}_j$$

$$+ \frac{1}{2} \sum_{q+1\leq i,j\leq n} F^3_{ij} dz_i \wedge d\overline{z}_j$$

is a real $(1,1)$*-form with constant coefficients on* $\mathbf{C}^n/\Gamma$ *such that* $F^3 := [F^3_{ij}]$ *is a real skew-symmetric matrix*$\}$

Then $E_{\mathbf{R}} \in \mathcal{E}^{1,1}$. *We have the following*

Theorem 3.1. *Let* $\mathbf{C}^n/\Gamma$ *be a toroidal group, where* Γ *is generated by* $\{e_1, \cdots, e_n, v_1, \cdots, v_q\}, V = [v_{ij}] = [v_1, \cdots, v_q], E = \frac{1}{2}\sum_{1\leq i,j\leq n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$ *and* $[E_{ij}] = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}$, *where* $E_1 \in \mathbf{Z}^{n\times n}$, *and* $E_3 \in \mathbf{Z}^{q\times q}$.
Then the following statements are equivalent.

(1) *There exists a line bundle* L *on* $\mathbf{C}^n/\Gamma$ *such that* $c_1(L) = E$.

(2) ${}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0$

(3) *There exists a real* $(1,1)$*-form* $E_{\mathbf{R}} \in \mathcal{E}^{1,1}$ *such that* E *is cohomologuous to* $E_{\mathbf{R}}$.

(4) *There exists a real* $(1,1)$*-form* $E_{\mathbf{R}} \in \mathcal{E}^{1,1}$ *such that*

$$E|T_\Gamma \times T_\Gamma = E_{\mathbf{R}}|T_\Gamma \times T_\Gamma. \tag{3.15}$$

When these hold, the real $(1,1)$*-form* $E_{\mathbf{R}} \in \mathcal{E}^{1,1}$ *is uniquely determined by* E.

Proof (1) $\Leftrightarrow$ (2) *This follows from* (3.10) *and* (3.11)

(3) $\Rightarrow$ (4) *By lemma 2.2, we have* 1*-form* $\psi^0 = \sum_{i=1}^{2n} \psi^0_i(t'')dt_i$ *such that* $E - E_{\mathbf{R}} = d\psi^0$. *Hence* $E|T_\Gamma \times T_\Gamma = E_{\mathbf{R}}|T_\Gamma \times T_\Gamma$.

(4) $\Rightarrow$ (3) *Let* $G = E - E_{\mathbf{R}}$. *Since* $E - E_{\mathbf{R}}$ *is d-closed* 2*-form, there exists a unique* $E' \in \mathbf{C}\{dt_1, \cdots, dt_{n+q}\}$ *and a* 1*-form* $\psi' = \sum_{i=1}^{2n} \psi_i'(t'')dt_i$ *such that* $E - E_{\mathbf{R}} = E' + d\psi'$. *Since* $d\psi'|T_\Gamma \times T_\Gamma = 0$, $E' \equiv 0$. *Hence* $E \sim E_{\mathbf{R}}$.
(2) $\Rightarrow$ (3) *It follows from* (3.12) *and* (3.14).
(3) $\Rightarrow$ (2) *Let* $F \in \mathcal{E}^{1,1}$ *be cohomologuous to* E. *Substituting* (2.7) *to* F, *by the uniqueness of* $F_{\mathbf{C}}$ *for* F, *we have* $F_{\mathbf{C}} = E_{\mathbf{C}}$. *Hence* $E^{0,2} = 0$ *and* (2) *holds.*
The uniqueness of $E_{\mathbf{R}}$. *In the proof of* (3) $\Rightarrow$ (2), *we see that the coefficients of* $F_{\mathbf{C}}$ *are completely determined by ones of* F. *Conversely the coefficients of* F *are completely determined by ones of* $F_{\mathbf{C}}$. *These correspondence is one to one similarly to those between* (3.12) *and* (3.14). *Since* $F_{\mathbf{C}} = E_{\mathbf{C}}$ *is uniquely defined by* E, F *is uniquely determined by* E.

We note that C.Vogt([7]) proved the equivalence of (1) *and* (2), *using the theory of multipliers for complex line bundles on* $\mathbf{C}^n/\Gamma$.
Let $E = \frac{1}{2} \sum_{1 \le i,j \le n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$ *and*
$E_{\mathbf{R}} = \sum_{1 \le i,j \le n} F_{ij} dz_i \wedge d\overline{z}_j \in \mathcal{E}^{1,1}$ *such that* $E \sim E_{\mathbf{R}}$. *We put*

$$H := 2\sqrt{-1}\ [F_{ij}]. \tag{3.16}$$

Since $[F_{ij}]$ *is skew-Hermitian,* H *is a Hermitian matrix. For* $\sigma = \sum_{i=1}^n \sigma_i \frac{\partial}{\partial z_i} \in T'$, *we put* $\widehat{\sigma} := \sigma + \overline{\sigma} = \sum_{i=1}^{2n} s_i \frac{\partial}{\partial t_i} \in T_{\mathbf{R}}$. *By* (1.9), *we can identify* σ *with* ${}^t(\sigma_1, \cdots, \sigma_n) \in \mathbf{C}^n$, *and* $\widehat{\sigma}$ *with* ${}^t(s_1, \cdots, s_{2n}) \in \mathbf{R}^{2n}$ *and* $E_{\mathbf{R}}$ *is a real skew-symmetric form on* $T_{\mathbf{R}}$ *and* $\mathbf{R}^{2n}$. *Put* $H(\sigma, \tau) := {}^t\sigma H \overline{\tau}$, *for* $\sigma, \tau \in \mathbf{C}^n$. *Then for any* $\sigma, \tau \in \mathbf{C}^n$, *we have*

$$ImH(\sigma, \tau) = E_{\mathbf{R}}(\widehat{\sigma}, \widehat{\tau}). \tag{3.17}$$

We set

$$\mathcal{H}^{1,1} := \{H; H \text{ is a Hermitian form on } T' \text{ such that } ImH \in \mathcal{E}^{1,1}\}$$

Since a Hermitian form is uniquely determined by its imaginary part, from theorem 3.1 we have the following

Theorem 3.2. *Let $\mathbf{C}^n/\Gamma$ be a toroidal group, where Γ is generated by $\{e_1, \cdots, e_n, v_1, \cdots, v_q\}$, and $E = \frac{1}{2}\sum_{1\leq i,j\leq n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$. Then the following statements are equivalent.*

(1) *There exists a line bundle L on $\mathbf{C}^n/\Gamma$ such that $c_1(L) = E$.*

(2) *There exists a Hermitian form $H \in \mathcal{H}^{1,1}$ on T' such that*

$$(3.18) \qquad ImH|T_\Gamma \times T_\Gamma = E|T_\Gamma \times T_\Gamma.$$

(3) *There exists a Hermitian form $H \in \mathcal{H}^{1,1}$ on $\mathbf{C}^n$ such that*

$$(3.19) \qquad ImH|\mathbf{R}_\Gamma \times \mathbf{R}_\Gamma = E|\mathbf{R}_\Gamma \times \mathbf{R}_\Gamma.$$

Further, when these statements hold, the Hermitian form $H \in \mathcal{H}^{1,1}$ which satisfies (3.18) or (3.19) is uniquely determined by E.

F. Capocasa and F. Catanese([1]) proved the existence of Hermitian form which satisfies (3.19) in theorem 3.2. Our result gives a characterization of such Hermitian forms which satisfy the uniqueness.

Let L be a complex line bundle on $\mathbf{C}^n/\Gamma$. Then L is defined by multipliers $\{e_\lambda(z) \in H^0(\mathbf{C}^n, \mathcal{O}^*); \lambda \in \Gamma\}$ for L which satisfy for any $z \in \mathbf{C}^n$

$$(3.20) \qquad e_{\lambda_1+\lambda_2}(z) = e_{\lambda_1}(z+\lambda_2)e_{\lambda_2}(z) \ \textit{for any} \ \ \lambda_1, \lambda_2 \in \Gamma.$$

For each $\lambda \in \Gamma$, there exists $f_\lambda(z) \in H^0(\mathbf{C}^n, \mathcal{O})$ such that $e_\lambda(z) = \exp(2\pi\sqrt{-1}f_\lambda(z))$. Let $E = c_1(L) \in H^2(\mathbf{C}^n/\Gamma, \mathbf{Z})$, then we have for any $\lambda_1, \lambda_2 \in \Gamma$([5]),

$$(3.21) \qquad E(\lambda_1, \lambda_2) = f_{\lambda_2}(z+\lambda_1) + f_{\lambda_1}(z) - f_{\lambda_1}(z+\lambda_2) - f_{\lambda_2}(z)$$

A complex line bundle is called a theta bundle if it is defined by the multipliers $\{\exp(a_\lambda(z)); a_\lambda(z)$ is a linear polynomial for $\lambda \in \Gamma\}$. A *map* $\alpha : \Gamma \to \mathbf{C}_1^* = \{z \in \mathbf{C}; |z| = 1\}$ *is called a semicharacter for E if it satisfies* $\alpha(\lambda_1 + \lambda_2) = \exp \pi\sqrt{-1}E(\lambda_1, \lambda_2)\alpha(\lambda_1)\alpha(\lambda_2)$ *for* $\lambda_1, \lambda_2 \in \Gamma$. *Let H be a Hermitian form satisfying the statement of* (2) *or* (3) *in theorem 3.2 for E and* α *a semicharacter for E. Put*

$$g_\lambda(z) := \alpha(\lambda)\exp(\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)) \tag{3.22}$$

Then $g_\lambda(z)$ *satisfies* (3.20). *Let* L_0 *be a complex line bundle on* $\mathbf{C}^n/\Gamma$ *defined by the multipliers* $\{g_\lambda(z); \lambda \in \Gamma\}$. *Then* L_0 *is a theta bundle and from* (3.21), *we have*

$$c_1(L_0) = ImH. \tag{3.23}$$

This means that $c_1(L_0) = c_1(L) = E$. *Hence we obtain the following*

Theorem 3.3 *Let L be a complex line bundle on* $\mathbf{C}^n/\Gamma$, *then there exist a theta bundle* L_0 *and a topologically trivial complex line budle* L_1 *on* $\mathbf{C}^n/\Gamma$ *such that* $L = L_0 \otimes L_1$

This theorem was first proved by C.Vogt[5]. We proved this theorem by constructing a theta bundle from the Hermitian form which is uniquely determined by L.

References

[1]. F. Capocasa and F. Catanese, Periodic meromorphic functions, Acta Math. **166** *(1991), 27–68.*

[2]. K. Kazama and T. Umeno, Complex abelian Lie groups with finite dimensional cohomology groups, J. Math. Soc. Japan **36** *(1984), 91–106.*

[3]. K. Kazama and T. Umeno, Some Dolbeault isomorphisms for locally trivial fiber spaces and applications, Proc. Japan Acad. **67** *(1991), 168–170.*

[4]. A. Morimoto, On the classification of non-compact complex abelian Lie groups, Trans. Amer. Math. Soc. **123** *(1966), 200–228.*

[5]. D. Mumford, Abelian Varieties, Oxford Univ. Press, 1970.

[6]. C. Vogt, Line bundles on toroidal groups, J. reine angew. Math. **335** *(1982), 197–215.*

[7]. C. Vogt, Two remarks concerning toroidal groups, manuscripta math. **41** *(1983), 217–232.*

Department of Mathematics
Kyushu Sangyo University
Fukuoka 813, Japan