DE RHAM COHOMOLOGY OF TOROIDAL GROUPS AND CHERN CLASSES OF THE COMPLEX LINE BUNDLES

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0. Introduction

Let $C^n/\Gamma$ be a toroidal group of complex dimension $n$, where $\Gamma$ is a discrete lattice of $C^n$ generated by $\mathbb{R}$-linearly independent vectors $e_1, \ldots, e_n, v_1, \ldots, v_q$ over $\mathbb{Z}$ and $e_i$ denotes the $i$-th unit vector of $C^n$.

C.Vogt([6]) characterized the toroidal group $C^n/\Gamma$ on which every complex line bundle is a theta bundle, investigating the theory of multipliers of complex line bundles on $C^n/\Gamma$. In particular, he proved that the finite dimensionality of the cohomology group $H^1(C^n/\Gamma, \mathcal{O})$ gives one of the characterizations.

On the other hand, we calculated the $\bar{\partial}$ -cohomology groups of $C^n/\Gamma$, using the Fourier expansions of $(r, s)$ -forms on $C^n/\Gamma([2],[3])$. In this paper, we shall apply these methods to the calculation of de Rham cohomology of $C^n/\Gamma$ and get several conditions for a $\mathbb{Z}$-valued skew-symmetric form $E$ on $\Gamma$ to be the Chern class of some complex line bundle on $C^n/\Gamma$. Further, we shall show the existence of some special class of hermitian forms which define complex line bundles and prove that the hermitian form is uniquely determined by a complex line bundle on $C^n/\Gamma$.

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1. Preliminaries

Throughout this paper we assume that $\mathbb{C}^n/\Gamma$ is a toroidal group of complex dimension $n$, where $\Gamma$ is a discrete lattice of $\mathbb{C}^n$ generated by $\mathbb{R}$-linearly independent vectors $\{e_1, \ldots, e_n, v_1 = t(v_{11}, \ldots, v_{n1}), \ldots, v_q = t(v_{1q}, \ldots, v_{nq})\}$ over $\mathbb{Z}$ and $e_i$ denotes the $i$-th unit vector of $\mathbb{C}^n$. We may assume $\det [\text{Im } v_{ij}; 1 \leq i, j \leq q] \neq 0$. We put $v_i = \sqrt{-1}e_i$ for $q + 1 \leq i \leq n$. Put

\begin{align}
V &= [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \ldots, v_q], \\
V_1 &= [v_{ij}; 1 \leq i, j \leq q], \text{ and } V_2 = [v_{ij}; q + 1 \leq i \leq n, 1 \leq j \leq q].
\end{align}

We set $K_{m,i} := \sum_{j=1}^{n} (m_jv_{ji} - m_{n+j})$ and $K_m := \max \{|K_{m,i}|; 1 \leq i \leq q\}$ for $m = (m_1, \ldots, m_{n+q}) \in \mathbb{Z}^{n+q}$. Since $\mathbb{C}^n/\Gamma$ is toroidal, $K_m > 0$ for any $m \in \mathbb{Z}^{n+q}\{0\}$\(^{(14)}\).

**Definition 1.1.** We say that a toroidal group $\mathbb{C}^n/\Gamma$ is of finite type if $\mathbb{C}^n/\Gamma$ satisfies the following condition:

There exists $a > 0$ such that

\[
\sup_{m \neq 0} \{\exp(-a\|m^*\|)/K_m\} < \infty, \quad \text{where } \|m^*\| = \max \{|m_i|; 1 \leq i \leq n\}.
\]

By the results of [3], a toroidal group $\mathbb{C}^n/\Gamma$ of finite type satisfies for $1 \leq r \leq n$,

\[
\dim H^s(\mathbb{C}^n/\Gamma, \Omega^r) = \begin{cases}
\begin{pmatrix} n \\ s \end{pmatrix} (\frac{q}{r})^s, & \text{if } 1 \leq s \leq q \\
0, & \text{if } s > q.
\end{cases}
\]

We put $\beta_i = \text{Im} v_i$ for $1 \leq i \leq n$ and $\beta = [\beta_{ij}] := [\beta_1, \ldots, \beta_n]$. Then $\beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{C}$ and we put $\gamma = [\gamma_{ij}] := \beta^{-1}$. For any $z \in \mathbb{C}^n$, we define two coordinates $z_1, \ldots, z_n$ and $t_1, \ldots, t_{2n}$ by

\begin{align}
z &= z_1\beta_1 + \cdots + z_n\beta_n \\
&= t_1e_1 + \cdots + t_ne_n + t_{n+1}v_1 + \cdots + t_{2n}v_n.
\end{align}
Then we have for $i = 1, \ldots, n$,

\begin{equation}
    t_i = \frac{1}{2\sqrt{-1}}(-\sum_{j=1}^{n} \bar{v}_{ij} z_j + \sum_{j=1}^{n} v_{ij} \bar{z}_j)
\end{equation}

and

\begin{equation}
    t_{n+i} = \frac{1}{2\sqrt{-1}}(z_i - \bar{z}_i).
\end{equation}

These coordinates $z = t(z_1, \ldots, z_n)$ and $t = t(t_1, \ldots, t_{2n})$ define local coordinates in $\mathbb{C}^n/\Gamma$. The mapping $\phi : \mathbb{C}^n \ni z = t(z_1, \ldots, z_n) \mapsto t = t(t_1, \ldots, t_{2n}) \in \mathbb{R}^{2n}$ induces an isomorphism as a real Lie group $\phi : \mathbb{C}^n/\Gamma \mapsto \mathbb{R}^{2n}/\phi(\Gamma) = \mathbb{T}^{n+q} \times \mathbb{R}^{n-q}$, where $\mathbb{T}^{n+q}$ is a real torus of real dimension $n + q$. For $t = t(t_1, \ldots, t_{2n}) \in \mathbb{R}^{2n}$ and $m = t(m_1, \ldots, m_{n+q}) \in \mathbb{Z}^{n+q}$, we put $t' = t(t_1, \ldots, t_{n+q}), t'' = t(t_{n+q+1}, \ldots, t_{2n})$ and $< m, t' > := m_1 t_1 + \cdots + m_{n+q} t_{n+q}$. Let $f$ be a complex valued $C^\infty$ function on $\mathbb{C}^n/\Gamma$. Then we have the Fourier expansion of $f$:

\begin{equation}
    f(t) = \sum_{m \in \mathbb{Z}^{n+q}} a^m(t'') \exp 2\pi \sqrt{-1} < m, t' > \quad \text{for} \quad t = \left(\begin{array}{c} t' \\ t'' \end{array}\right) \in \mathbb{R}^{2n}.
\end{equation}

By the standard argument of Fourier analysis, a series

\[ \sum_{m \in \mathbb{Z}^{n+q}} a^m(t'') \exp 2\pi \sqrt{-1} < m, t' > \]

converges to a $C^\infty$ function on $\mathbb{C}^n/\Gamma$ if and only if

\begin{equation}
    C(\ell, I, R) := \sup_{|t''| \leq R} \left\{ \left| \frac{\partial^\ell a^m(t'')}{\partial t''^{\ell}} \right| \|m\|^{\ell}; m \in \mathbb{Z}^{n+q} \right\} < \infty,
\end{equation}

for any positive integers $\ell, I$ and any positive number $R$, where $|t''| = \sqrt{t_{n+q+1}^2 + \cdots + t_{2n}^2}$ and $\|m\| = \max\{|m_i|; i = 1, \ldots, n + q\}$.

Let $T' := \mathbb{C} \left\{ \frac{\partial}{\partial z_i}; i = 1, \ldots, n \right\}$ be the holomorphic tangent space of $\mathbb{C}^n/\Gamma$ at 0,

$T_R := \mathbb{R} \left\{ \frac{\partial}{\partial t_i}; i = 1, \ldots, 2n \right\}$ the real tangent space of $\mathbb{C}^n/\Gamma$ at 0,

$T_T := \mathbb{R} \left\{ \frac{\partial}{\partial t_i}; i = 1, \ldots, n + q \right\}$, and $R_T := \mathbb{R} \{e_1, \ldots, e_n, v_1, \ldots, v_q\}$. 
For \( \sigma = \sum_{i=1}^{n} \sigma_i \frac{\partial}{\partial z_i} \in T' \), we put \( \hat{\sigma} := \sigma + \bar{\sigma} = \sum_{i=1}^{2n} \bar{s}_i \frac{\partial}{\partial \bar{t}_i} \in T_\mathbb{R} \). Then the mapping

\[ T' \ni \sigma \mapsto \hat{\sigma} \in T_\mathbb{R} \]

is an \( \mathbb{R} \)-isomorphism.

From (1.3) and (1.4), we have

\[ \sigma_1 \beta_1 + \cdots + \sigma_n \beta_n = s_1 e_1 + \cdots + s_n e_n + s_{n+1} v_1 + \cdots + s_{2n} v_n. \]

By the mappings

\[ T' \ni \sigma \mapsto \iota(\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^n \quad \text{and} \]

\[ T_\mathbb{R} \ni \hat{\sigma} \mapsto \iota(s_1, \ldots, s_{2n}) \in \mathbb{R}^{2n}, \]

we can identify \( T' \) with \( \mathbb{C}^n \), \( T_\mathbb{R} \) with \( \mathbb{R}^{2n} \) and \( T_\Gamma \) with \( \mathbb{R}_\Gamma \), respectively.

2. de Rham cohomology of toroidal groups

In this section, we calculate the de Rham cohomology groups of toroidal groups \( \mathbb{C}^n / \Gamma \). Let \( \mathcal{C} \) be the sheaf of germs of complex valued \( \mathcal{C}^\infty \) functions on \( \mathbb{C}^n / \Gamma \), \( \mathcal{C}^p \) the sheaf of germs of \( \mathcal{C}^\infty \) \( p \)-forms on \( \mathbb{C}^n / \Gamma \), and \( \Omega^r \) the sheaf of germs of holomorphic \( r \)-forms on \( \mathbb{C}^n / \Gamma \). We denote by \( Z_d(\mathbb{C}^n / \Gamma, \mathcal{C}^p) \) the space of \( d \)-closed \( \mathcal{C}^\infty \) \( p \)-forms on \( \mathbb{C}^n / \Gamma \) and by \( B_d(\mathbb{C}^n / \Gamma, \mathcal{C}^p) \) the space of \( d \)-exact \( \mathcal{C}^\infty \) \( p \)-forms on \( \mathbb{C}^n / \Gamma \). We have

\[ H^p(\mathbb{C}^n / \Gamma, \mathcal{C}) = \frac{Z_d(\mathbb{C}^n / \Gamma, \mathcal{C}^p)}{B_d(\mathbb{C}^n / \Gamma, \mathcal{C}^p)}. \]

Let \( \varphi \) be a \( \mathcal{C}^\infty \) \( p \)-form on \( \mathbb{C}^n / \Gamma \), we write

\[ \varphi(t) = \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq 2n} \varphi_{i_1 \ldots i_p}(t) dt_{i_1} \wedge \cdots \wedge dt_{i_p}. \]

We expand \( \varphi_{i_1 \ldots i_p}(t) \) as in (1.5) and put
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\( \varphi_{t_1 \cdots t_p}(t) = \sum_{m \in \mathbb{Z}^{n+q}} a^m_{i_1 \cdots i_p}(t'') \exp 2\pi \sqrt{-1} < m, t' > \) and

\[
\varphi^m := \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p \leq 2n} a^m_{i_1 \cdots i_p}(t'') \exp 2\pi \sqrt{-1} < m, t' >
\]

where \( t' = dt_{i_1} \wedge \cdots \wedge dt_{i_p} \).

Then \( \varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m \). Suppose \( \varphi \in B_d(C^n/\Gamma, C_p) \). There exists a \( C^\infty(p-1) \)-form \( \psi = \sum_{m \in \mathbb{Z}^{n+q}} \psi^m \) such that \( \varphi = \overline{\partial} \psi \). Then we have \( \varphi^m = \overline{\partial} \psi^m \) for any \( m \in \mathbb{Z}^{n+q} \). We put

\[
\psi^m = \frac{1}{(p-1)!} \sum_{1 \leq i_1, \cdots, i_p-1 \leq 2n} b^m_{i_1 \cdots i_p-1}(t'') \exp 2\pi \sqrt{-1} < m, t' >
\]

where \( t' = dt_{i_1} \wedge \cdots \wedge dt_{i_p} \).

The equation \( \varphi = \overline{\partial} \psi \) implies for any \( m \in \mathbb{Z}^{n+q} \) and \( 1 \leq i_1 < \cdots < i_p \leq 2n \),

\[
a^m_{i_1 \cdots i_p}(t'') = \sum_{k=1}^{\ell} \frac{(-1)^k}{(k-1)!} (2\pi \sqrt{-1} m_k) b^m_{i_1 \cdots i_k \cdots i_p}(t'')
\]

\[
+ \sum_{k=\ell+1}^{p} \frac{(-1)^k}{(k-1)!} \partial b^m_{i_1 \cdots i_k \cdots i_p}(t'') \bigg/ \partial t_{i_k} \tag{2.1}
\]

where \( \ell := \max\{k; i_k \leq n + q\} \). In particular, we have

\[
1 \leq i_1, \cdots, i_p \leq n + q \Rightarrow a^0_{i_1 \cdots i_p}(t'') \equiv 0. \tag{2.2}
\]

Now suppose \( \varphi \in Z_d(C^n/\Gamma, C_p) \). For each \( m = (m_1, \cdots, m_n+q) \in \mathbb{Z}^{n+q} \setminus \{0\} \) we put \( \iota(m) := \max\{i; m_i \neq 0\} \) and \( M(m) := m_{\iota(m)} \). For any \( 1 \leq i_1, \cdots, i_p \leq 2n \) and \( m \in \mathbb{Z} \setminus \{0\} \), we have
From (2.3) and (2.4), we have \( \varphi^m = d\psi^m \), for any \( m \in \mathbb{Z}^{n+q} \setminus \{0\} \).

Further, from (1.6) and (2.4), \( \tilde{\psi} := \sum_{m \in \mathbb{Z}^{n+q} \setminus \{0\}} \psi^m \) converges in \( H^0 (\mathbb{C}^n/\Gamma, C^p) \). Hence we have the following

**Lemma 2.1.** Let \( \varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m \) be a \( C^\infty \)-closed \( p \)-form on \( \mathbb{C}^n/\Gamma \).

Then we have a \( C^\infty (p-1) \)-form \( \tilde{\psi} = \sum_{m \in \mathbb{Z}^{n+q} \setminus \{0\}} \psi^m \) defined by (2.4) satisfying \( \varphi = \varphi^0 + d\tilde{\psi} \).

In case \( m = 0 \) we get the following

**Lemma 2.2.** Let \( \varphi^0 = \frac{1}{p!} \sum_{1 \leq t_1, \ldots, t_p \leq 2n} a^0_{t_1 \ldots t_p} (t'') dt_{t_1} \wedge \cdots \wedge dt_{t_p} \)

be a \( C^\infty \)-closed \( p \)-form on \( \mathbb{C}^n/\Gamma \). Then there exists a unique \( p \)-form with constant coefficients \( \chi = \frac{1}{p!} \sum_{1 \leq t_1, \ldots, t_p \leq n+q} c_{t_1 \ldots t_p} dt_{t_1} \wedge \cdots \wedge dt_{t_p} \)
and a \((p-1)\)-form \(\psi^0 = \frac{1}{(p-1)!}\sum_{1 \leq i_1, \ldots, i_{p-1} \leq 2n} b^0_{i_1 \ldots i_{p-1}} (t''')dt_{i_1} \wedge \cdots \wedge dt_{i_{p-1}}\) satisfying \(\varphi^0 = \chi + d\psi^0\).

**Proof.** The uniqueness of \(\chi\) immediately follows by (2.2). We shall show the existence of \(\psi^0\).

For each \(1 \leq i_1 < \cdots < i_{p+1} \leq 2n\), we put \(\ell := \max\{k; i_k \leq n+q\}\). We have

\[
\sum_{k=\ell+1}^{p+1} (-1)^{k+1} \frac{\partial a^0_{t_1 \ldots i_{k+1} \ldots i_{p+1}} (t''')}{\partial t_{i_k}} = 0
\]

In case \(\ell = p\), for each \(1 \leq i_1 < \cdots < i_p \leq n+q\) and \(n+q \leq \iota \leq 2n\), we have \(\frac{\partial a^0_{t_1 \ldots i_p} (t''')}{\partial t_{i_k}} = 0\). Hence \(c_{t_1 \ldots i_p} := a^0_{t_1 \ldots i_p} (t''')\) are constant. Put

\[
\chi := \frac{1}{p!} \sum_{1 \leq t_1, \ldots, t_p \leq n+q} c_{t_1 \ldots i_p} dt_{t_1} \wedge \cdots \wedge dt_{i_p}.
\]

In case \(\ell < p\), for each \(1 \leq t_1 < \cdots < t_\ell \leq n+q\),

\[
\varphi^0_{t_1 \ldots t_\ell} := \sum_{n+g+1 \leq t_{\ell+1} < \cdots < t_p \leq 2n} a^0_{t_1 \ldots t_{\ell+1} \ldots i_p} (t''')dt_{t_{\ell+1}} \wedge \cdots \wedge dt_{i_p}
\]

is d-closed \(p'\)-form in \(\mathbb{R}^{n-q}\), where \(p' = p - \ell\). Then we have \((p' - 1)\)-form on \(\mathbb{R}^{n-q}\)

\[
\psi^0_{t_1 \ldots t_\ell} := \sum_{n+g+1 \leq t_{\ell+1} < \cdots < t_p \leq 2n} b^0_{t_1 \ldots t_{\ell+1} \ldots i_p} (t''')dt_{t_{\ell+1}} \wedge \cdots \wedge dt_{i_p}
\]

satisfying \(d\psi^0_{t_1 \ldots t_\ell} = \varphi^0_{t_1 \ldots t_\ell}\). Put

\[
\psi^0 := \sum_{\ell=0}^{p-1} (-1)^\ell \sum_{1 \leq t_1 < \cdots < t_\ell \leq n+q} b^0_{t_1 \ldots t_{\ell+1} \ldots i_p} (t''')dt_{t_1} \wedge \cdots \wedge dt_{i_p}.
\]

We have \(d\psi^0 = \varphi^0 - \chi\)

Q.E.D.
Summarizing lemma 2.1 and lemma 2.2, we have the following.

**Proposition 2.1.** Let $\varphi$ be a $C^\infty d$-closed $p$-form on a toroidal group $C^n/\Gamma$. Then there exists a unique $p$-form with constant coefficients

(2.5) \[ \chi = \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p \leq n+q} c_{i_1, \cdots, i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p} \]

and a $C^\infty(p-1)$-form $\psi$ on $C^n/\Gamma$ satisfying $\varphi = \chi + d\psi$.

**Notation** For any $C^\infty d$-closed $p$-forms $\varphi_1$ and $\varphi_2$ on $C^n/\Gamma$ we write $\varphi_1 \sim \varphi_2$ when $\varphi_1$ and $\varphi_2$ are cohomologous, namely, there is a $C^\infty(p-1)$-form $\psi$ on $C^n/\Gamma$ such that $\varphi_1 - \varphi_2 = d\psi$.

Since $t_{n+i}$ are global functions for $i = q+1, \cdots, 2n$ and from (1.4), we have

(2.6) \[ dt_i \sim \frac{1}{2\sqrt{-1}} \left( - \sum_{j=1}^q \overline{u}_{ij} d z_j + \sum_{j=1}^q v_{ij} d \overline{z}_j \right) \text{ for } i = 1, \cdots, q, \]

\[ dt_i \sim \frac{1}{2\sqrt{-1}} \left( - \sum_{j=1}^q \overline{u}_{ij} d z_j + 2\sqrt{-1} d z_i + \sum_{j=1}^q v_{ij} d \overline{z}_j \right) \]

\[ \text{ for } i = q+1, \cdots, n, \]

\[ dt_{n+i} \sim \frac{1}{2\sqrt{-1}} \left( d z_i - d \overline{z}_i \right) \text{ for } i = 1, \cdots, q, \text{ and } \]

(2.7) \[ d z_i \sim d \overline{z}_i \text{ for } i = q+1, \cdots, n. \]

Conversely it is easy to show that $d z_i, d \overline{z}_j$, for $i = 1, \cdots, n$ and $j = 1, \cdots, q$ are cohomologous to linear combinations of $dt_1, \cdots, dt_{n+q}$. Substituting (2.6) to (2.5), we get

(2.8) \[ \chi \sim \chi C := \sum_{r+s=p} \frac{1}{r!s!} \sum_{1 \leq i_1, \cdots, i_r, j_1, \cdots, j_s \leq n} c_{i_1, \cdots, i_r, j_1, \cdots, j_s} d z_{i_1} \wedge \cdots \wedge d z_{i_r} \wedge d \overline{z}_{j_1} \wedge \cdots \wedge d \overline{z}_{j_s}, \]

where $\chi C$ is a constant $p$-form on $C^n/\Gamma$. From (2.6), the mapping

(2.9) \[ \bigwedge^p C\{dt_1, \cdots, dt_{n+q}\} \ni \chi \mapsto \chi C \in \bigwedge^p C\{d z_1, \cdots, d z_n, d \overline{z}_1, \cdots, d \overline{z}_q\} \]
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is one to one correspondence. Hence we get the following

**Theorem 2.1.** Let $C^n/\Gamma$ be a toroidal group where $\Gamma$ is generated by
\{e_1, \cdots, e_n, v_1, \cdots, v_q\}. Then we have:

1. any cohomology class of $\varphi$ in $Z_d(C^n/\Gamma, \mathbb{C}^\rho)$ is represented by
constant forms $\chi \in \bigwedge^P C\{dt_1, \cdots, dt_{n+q}\}$ with respect to the basis
\{\{dt_1, \cdots, dt_{n+q}\} and $\chi_C \in \mathbb{C}\{dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_q\}$ with respect to the basis
\{\{dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_q\}. Further these forms $\chi$ and $\chi_C$ are uniquely
determined by $\varphi$.

2. $H^p(C^n/\Gamma, \mathbb{C}) \cong \bigwedge^P C\{dt_1, \cdots, dt_{n+q}\}$
   \[\cong \bigwedge^P C\{dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_q\}\]
   for $1 \leq p \leq n + q$
   \[= 0 \text{ for } p \geq n + q + 1.\]

In (2.8), we put

$$\chi^{r,s} := \frac{1}{r! s!} \sum_{1 \leq t_1, \cdots, t_r \leq n} \sum_{1 \leq j_1, \cdots, j_s \leq q} c_{t_1, \cdots, t_r} dz_{t_1} \wedge \cdots \wedge dz_{t_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

for $0 \leq r \leq n$, and $0 \leq s \leq q$. Since $\chi^{r,s}$ is $\overline{\partial}$-closed and from theorem
2.1 we get homomorphisms

(2.10) $\Phi^{r,s} : H^p(C^n/\Gamma, \mathbb{C}) \ni [\chi] \longmapsto [\chi^{r,s}] \in H^s(C^n/\Gamma, \Omega^r)$

for $0 \leq r \leq n$ and $0 \leq s \leq q$ such that $r + s = p$. In case $C^n/\Gamma$ is a
toroidal group of finite type, by [3]

$$H^s(C^n/\Gamma, \Omega^r) \cong$$

$$\mathbb{C}\{dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s} ;$$

$1 \leq i_1 < \cdots < i_r \leq n, 1 \leq j_1 < \cdots < j_s \leq q\}$

Thus we have the following
Theorem 2.2 Let $\mathbb{C}^n/\Gamma$ be a toroidal group of finite type where $\Gamma$ is generated by $\{e_1, \ldots, e_n, v_1, \ldots, v_q\}$. Then the homomorphisms $\Phi^{r,s}$ defined by (2.10) are onto for $0 \leq r \leq n$ and $0 \leq s \leq q$ such that $r+s = p$. Further we get a Hodge decomposition

$$H^p(\mathbb{C}^n/\Gamma, \mathbb{C}) \cong \bigoplus_{0 \leq r \leq n, 0 \leq s \leq q} H^q(\mathbb{C}^n/\Gamma, \Omega^r).$$

We note C. Vogt([7]) also showed the Hodge decomposition of theorem 2.2 by comparing the complex dimensions of the above cohomology spaces.

3. Chern classes of complex line bundles over toroidal groups

In this section we shall study the condition for $E \in H^2(\mathbb{C}^n/\Gamma, \mathbb{Z})$ to be the Chern class of some complex line bundle $L$ on $\mathbb{C}^n/\Gamma$, and describe $L$ by $E$.

We put $\Gamma = \mathbb{Z}\{e_1, \ldots, e_n, v_1, \ldots, v_q\} = \mathbb{Z}\{u_1, \ldots, u_{n+q}\}$. We denote by $\hat{u}_i \in H_1(\mathbb{C}^n/\Gamma, \mathbb{Z})$ the loop with base point $[0] \in \mathbb{C}^n/\Gamma$ lifts to a path in $\mathbb{C}^n$ starting at $0$ and ending at a point $u_i$, for each $i$. Since $\int_{\hat{u}_i} dt_j = \delta_{ij}$ for $1 \leq i, j \leq n + q$, we have

$$H^1(\mathbb{C}^n/\Gamma, \mathbb{Z}) \cong \mathbb{Z}\{dt_1, \ldots, dt_{n+q}\}, \text{ and}$$

$$H^p(\mathbb{C}^n/\Gamma, \mathbb{Z}) \cong \bigwedge^p \mathbb{Z}\{dt_1, \ldots, dt_{n+q}\}.$$  

Let $E \in H^2(\mathbb{C}^n/\Gamma, \mathbb{Z})$, then we can write

$$E = \frac{1}{2} \sum_{1 \leq i, j \leq n+q} E_{ij} dt_i \wedge dt_j,$$

where $[E_{ij}]$ is a $\mathbb{Z}$-valued skew-symmetric matrix. We denote by $\mathbb{Z}^{n \times n}$ (resp. $\mathbb{C}^{n \times n}$) the set of $n \times n$, $\mathbb{Z}$ (resp. $\mathbb{C}$)-valued matrices. We put

$$[E_{ij}] = \begin{bmatrix} E_1 & E_2 \\ -iE_2 & E_3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} F_1 & F_2 \\ -iF_2 & F_3 \end{bmatrix}, \text{ and}$$

$$E_2 = \begin{bmatrix} F_4 \\ F_5 \end{bmatrix}.$$
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where $E_1 \in \mathbb{Z}^{n \times n}$, $E_2, F_1 \in \mathbb{Z}^{q \times q}$, $F_3 \in \mathbb{Z}^{(n-q) \times (n-q)}$, and $F_4 \in \mathbb{Z}^{q \times (n-q)}$.

From theorem 2.1, we have a unique constant 2-form

$$E_\mathcal{C} \in \bigwedge^2 \mathcal{C} \{dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n\}$$

such that $E \sim E_\mathcal{C}$ in $Z_d(C^n/\Gamma, \mathcal{C})$.

Substituting (2.6) to (3.3), we get

(3.5) $$E_\mathcal{C} = E^{2,0} + E^{1,1} + E^{0,2},$$

where $E^{r,s} = \Phi^{r,s}(E)$.

We put

(3.6) $$E^{2,0} = \frac{1}{2} \sum_{1 \leq i, j \leq n} A_{ij} dz_i \wedge dz_j, \quad E^{1,1} = \sum_{1 \leq i \leq n \atop 1 \leq j \leq q} B_{ij} dz_i \wedge d\bar{z}_j \quad \text{and}$$

$$E^{0,2} = \frac{1}{2} \sum_{1 \leq i, j \leq q} C_{ij} d\bar{z}_i \wedge d\bar{z}_j.$$

Put

(3.7) $$A = [A_{ij}] = \left[ \begin{array}{cc} A_1 & A_2 \\ -tA_2 & A_3 \end{array} \right], \quad B = [B_{ij}] = \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right], \quad \text{and} \quad C = [C_{ij}],$$

where $A_1, B_1 \in \mathbb{C}^{q \times q}, A_3 \in \mathbb{C}^{(n-q) \times (n-q)}$. Then we have

(3.8) $$A_1 = -\frac{1}{4} (t \overline{V} E_1 \overline{V} + t E_2 \overline{V} - t \overline{V} E_2 + E_3),$$

$$A_2 = \frac{\sqrt{-1}}{2} (t \overline{V} F_2 + t \overline{V} F_3 + t F_5),$$

$$A_3 = F_3, \quad B_1 = \frac{1}{4} (t \overline{V} E_1 V + t E_2 V - t \overline{V} E_2 + E_3),$$

$$B_2 = \frac{\sqrt{-1}}{2} (t F_2 V_1 - F_3 V_2 + F_5) = -t \overline{A_2},$$

$$C = -\frac{1}{4} (t \overline{V} E_1 V + t E_2 V - t \overline{V} E_2 + E_3) = \overline{A_1}.$$

By the exact sequence

$$H^1(C^n/\Gamma, \mathcal{O}^*) \xrightarrow{c_1} H^2(C^n/\Gamma, \mathcal{O}) \xrightarrow{\cdot} H^2(C^n/\Gamma, \mathcal{O}),$$

for any $E \in H^2(C^n/\Gamma, \mathcal{O})$, there exists a line bundle $L \in H^1(C^n/\Gamma, \mathcal{O}^*)$ such that $c_1(L) = E$ if and only if
(3.9) \( \nu(E) = 0 \) in \( H^2(\mathbb{C}^n/\Gamma, \mathcal{O}) \).

From (2.10), (3.9) is equivalent to

(3.10) \( E^{0,2} \) is \( \overline{\partial} \)-exact.

Since \( E^{0,2} \) is a constant form, from the lemma 2.1 of [2], (3.10) is equivalent to

(3.11) \( \iota \bar{V}E_1V + \iota \bar{E}_2V - \iota \bar{V}E_2 + E_3 = 0 \)

From (3.6), (3.8) and (3.11), we obtain

(3.12) \[
E_C = \sum_{1 \leq i, j \leq q} A_{ij} dz_i \wedge dz_j + \frac{1}{2} \sum_{q+1 \leq i, j \leq n} E_{ij} dz_i \wedge d\bar{z}_j
\]

\[
+ \sum_{1 \leq i, j \leq q} B_{ij} dz_i \wedge d\bar{z}_j - \sum_{q+1 \leq i, j \leq n} \overline{A}_{ji} dz_i \wedge d\bar{z}_j,
\]

Further,

(3.13) \( B_1 = [B_{ij}; 1 \leq i, j \leq q] \) is a skew-Hermitian matrix and

\( F_3 = [E_{ij}; q+1 \leq i, j \leq n] \) is a real skew-symmetric matrix.

From (2.7), \( E_C \) is cohomologous to

(3.14) \[
E_R := \sum_{1 \leq i, j \leq q} B_{ij} dz_i \wedge d\bar{z}_j + \sum_{1 \leq i \leq q} A_{ij} dz_i \wedge d\bar{z}_j
\]

\[
- \sum_{q+1 \leq i \leq n} \overline{A}_{ji} dz_i \wedge d\bar{z}_j + \frac{1}{2} \sum_{q+1 \leq i, j \leq n} E_{ij} dz_i \wedge d\bar{z}_j.
\]

From (3.13), we see that \( E_R \) is a real \((1,1)\)-form with constant coefficients on \( \mathbb{C}^n/\Gamma \). We set
\[ E_{1,1} := \{ F; F = \sum_{1 \leq i, j \leq q} F_{ij}^1 dz_i \wedge d\bar{z}_j + \sum_{1 \leq i \leq q \leq j \leq n} F_{ij}^2 dz_i \wedge d\bar{z}_j \]

\[ - \sum_{q+1 \leq i \leq n \leq j \leq q} F_{ij}^2 dz_i \wedge d\bar{z}_j \]

\[ + \frac{1}{2} \sum_{q+1 \leq i \leq j \leq n} F_{ij}^3 dz_i \wedge d\bar{z}_j \]

is a real \((1, 1)\)-form with constant coefficients on \( \mathbb{C}^n / \Gamma \) such that \( F^3 := [F_{ij}^3] \) is a real skew-symmetric matrix.

Then \( E_R \in E_{1,1} \). We have the following

**Theorem 3.1.** Let \( \mathbb{C}^n / \Gamma \) be a toroidal group, where \( \Gamma \) is generated by \( \{ e_1, \ldots, e_n, v_1, \ldots, v_q \} \), \( V = [v_{ij}] = [v_1, \ldots, v_q] \), \( E = \frac{1}{2} \sum_{1 \leq i, j \leq n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbb{C}^n / \Gamma, \mathbb{Z}) \) and \( [E_{ij}] = \left[ \begin{array}{cc} E_1 & E_2 \\ -iE_2 & E_3 \end{array} \right] \), where \( E_1 \in \mathbb{Z}^{n \times n} \), and \( E_3 \in \mathbb{Z}^{q \times q} \).

Then the following statements are equivalent.

1. There exists a line bundle \( L \) on \( \mathbb{C}^n / \Gamma \) such that \( c_1(L) = E \).
2. \( t^*V + \frac{i}{2}E_1 + E_2V - E_3 = 0 \)
3. There exists a real \((1, 1)\)-form \( E_R \in E_{1,1} \) such that \( E \) is cohomologous to \( E_R \).
4. There exists a real \((1, 1)\)-form \( E_R \in E_{1,1} \) such that

\[(3.15) \quad E|_{T_\Gamma \times T_\Gamma} = E_R|_{T_\Gamma \times T_\Gamma}.

When these hold, the real \((1, 1)\)-form \( E_R \in E_{1,1} \) is uniquely determined by \( E \).

**Proof** (1) \( \leftrightarrow \) (2) This follows from (3.10) and (3.11)

(3) \( \Rightarrow \) (4) By lemma 2.2, we have 1-form \( \psi^0 = \sum_{i=1}^{2n} \psi^0_i (t^i) dt_i \) such that \( E - E_R = d\psi^0 \). Hence \( E|_{T_\Gamma \times T_\Gamma} = E_R|_{T_\Gamma \times T_\Gamma} \).
(4) \( \Rightarrow \) (3) Let \( G = E - E_R \). Since \( E - E_R \) is d-closed 2-form, there exists a unique \( E' \in C\{dt_1, \ldots, dt_{n+q}\} \) and a 1-form \( \phi' = \sum_{i=1}^{2n} \psi'_i(t'') dt_i \)

such that \( E - E_R = E' + d\phi' \). Since \( d\phi'|T_1 \times T_1 = 0 \), \( E' \equiv 0 \). Hence \( E \sim E_R \).

(2) \( \Rightarrow \) (3) It follows from (3.12) and (3.14).

(3) \( \Rightarrow \) (2) Let \( F \in \mathcal{E}^{1,1} \) be cohomologuous to \( E \). Substituting (2.7) to \( F \), by the uniqueness of \( F \) for \( F \in \mathcal{E}^{1,1} \), we have \( F \equiv E \). Hence \( E^{0,2} = 0 \) and (2) holds.

The uniqueness of \( E_R \). In the proof of (3) \( \Rightarrow \) (2), we see that the coefficients of \( F \) are completely determined by ones of \( E \). Conversely the coefficients of \( F \) are completely determined by ones of \( F \). These correspondences are one to one similar to those between (3.12) and (3.14). Since \( F \equiv E \) is uniquely defined by \( E \), \( F \) is uniquely determined by \( E \).

We note that C.Vogt ([7]) proved the equivalence of (1) and (2), using the theory of multipliers for complex line bundles on \( \mathbb{C}^n / \Gamma \).

Let \( E = \frac{1}{2} \sum_{1 \leq i, j \leq n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbb{C}^n / \Gamma, \mathbb{Z}) \) and

\[
E_R = \sum_{1 \leq i, j \leq n} F_{ij} dz_i \wedge d\bar{z}_j \in \mathcal{E}^{1,1} \text{ such that } E \sim E_R.
\]

We put

\[
(3.16) \quad H := 2\sqrt{-1} [F_{ij}].
\]

Since \( [F_{ij}] \) is skew-Hermitian, \( H \) is a Hermitian matrix. For \( \sigma = \sum_{i=1}^{n} \sigma_i \frac{\partial}{\partial z_i} \in T' \), we put \( \widehat{\sigma} := \sigma + \bar{\sigma} = \sum_{i=1}^{2n} s_i \frac{\partial}{\partial t_i} \in T_R \). By (1.9), we can identify \( \sigma \) with \( (\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^n \), and \( \widehat{\sigma} \) with \( (s_1, \ldots, s_{2n}) \in \mathbb{R}^{2n} \) and \( E_R \) is a real skew-symmetric form on \( T_R \) and \( \mathbb{R}^{2n} \). Put \( H(\sigma, \tau) := \sigma \tau \), for \( \sigma, \tau \in \mathbb{C}^n \). Then for any \( \sigma, \tau \in \mathbb{C}^n \), we have

\[
(3.17) \quad \text{Im} H(\sigma, \tau) = E_R(\widehat{\sigma}, \widehat{\tau}).
\]

We set

\[
\mathcal{H}^{1,1} := \{ H; H \text{ is a Hermitian form on } T' \text{ such that Im} H \in \mathcal{E}^{1,1} \}
\]
Since a Hermitian form is uniquely determined by its imaginary part, from theorem 3.1 we have the following

**Theorem 3.2.** Let $\mathbb{C}^n/\Gamma$ be a toroidal group, where $\Gamma$ is generated by $\{e_1, \ldots, e_n, v_1, \ldots, v_q\}$, and $E = \frac{1}{2} \sum_{1 \leq i, j \leq n+q} E_{ij} dt_i \wedge dt_j \in H^2(\mathbb{C}^n/\Gamma, \mathbb{Z})$. Then the following statements are equivalent.

1. There exists a line bundle $L$ on $\mathbb{C}^n/\Gamma$ such that $c_1(L) = E$.
2. There exists a Hermitian form $H \in \mathcal{H}^{1,1}$ on $T^\ast$ such that

\[ \text{Im } H | T^\ast \times T^\ast = E | T^\ast \times T^\ast. \]  

(3.18)  

3. There exists a Hermitian form $H \in \mathcal{H}^{1,1}$ on $\mathbb{C}^n$ such that

\[ \text{Im } H | R^\ast \times R^\ast = E | R^\ast \times R^\ast. \]  

(3.19)  

Further, when these statements hold, the Hermitian form $H \in \mathcal{H}^{1,1}$ which satisfies (3.18) or (3.19) is uniquely determined by $E$.

F. Capocasa and F. Catanese([1]) proved the existence of Hermitian forms which satisfies (3.19) in theorem 3.2. Our result gives a characterization of such Hermitian forms which satisfies the uniqueness.

Let $L$ be a complex line bundle on $\mathbb{C}^n/\Gamma$. Then $L$ is defined by multipliers $\{e_\lambda(z) \in H^0(\mathbb{C}^n, \mathcal{O}); \lambda \in \Gamma\}$ for $L$ which satisfy for any $z \in \mathbb{C}^n$

\[ e_{\lambda_1 + \lambda_2}(z) = e_{\lambda_1}(z + \lambda_2) e_{\lambda_2}(z) \text{ for any } \lambda_1, \lambda_2 \in \Gamma. \]  

(3.20)  

For each $\lambda \in \Gamma$, there exists $f_\lambda(z) \in H^0(\mathbb{C}^n, \mathcal{O})$ such that $e_\lambda(z) = \exp(2\pi \sqrt{-1} f_\lambda(z))$. Let $E = c_1(L) \in H^2(\mathbb{C}^n/\Gamma, \mathbb{Z})$, then we have for any $\lambda_1, \lambda_2 \in \Gamma([5])$,

\[ E(\lambda_1, \lambda_2) = f_{\lambda_2}(z + \lambda_1) + f_{\lambda_1}(z) - f_{\lambda_1}(z + \lambda_2) - f_{\lambda_2}(z) \]  

(3.21)  

A complex line bundle is called a theta bundle if it is defined by the multipliers $\{\exp(\alpha_\lambda(z)); \alpha_\lambda(z) \text{ is a linear polynomial for } \lambda \in \Gamma\}$. A
map $\alpha : \Gamma \to C^*_1 = \{z \in C; |z| = 1\}$ is called a semicharacter for $E$ if it satisfies $\alpha(\lambda_1 + \lambda_2) = \exp \pi \sqrt{-1}E(\lambda_1, \lambda_2)\alpha(\lambda_1)\alpha(\lambda_2)$ for $\lambda_1, \lambda_2 \in \Gamma$. Let $H$ be a Hermitian form satisfying the statement of (2) or (3) in theorem 3.2 for $E$ and $\alpha$ a semicharacter for $E$. Put

$$
(3.22) \quad g_\lambda(z) := \alpha(\lambda) \exp(\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))
$$

Then $g_\lambda(z)$ satisfies (3.20). Let $L_0$ be a complex line bundle on $C^n/\Gamma$ defined by the multipliers $\{g_\lambda(z); \lambda \in \Gamma\}$. Then $L_0$ is a theta bundle and from (3.21), we have

$$
(3.23) \quad c_1(L_0) = \text{Im}H.
$$

This means that $c_1(L_0) = c_1(L) = E$. Hence we obtain the following

Theorem 3.3 Let $L$ be a complex line bundle on $C^n/\Gamma$, then there exist a theta bundle $L_0$ and a topologically trivial complex line bundle $L_1$ on $C^n/\Gamma$ such that $L = L_0 \otimes L_1$

This theorem was first proved by C.Vogt[5]. We proved this theorem by constructing a theta bundle from the Hermitian form which is uniquely determined by $L$.

References


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