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BLOCH FUNCTIONS AND THE BLOCH NUMBER

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1. Introduction

Let Ω be a hyperbolic region in the complex plane $\mathbf{C}$ and $\lambda_\Omega(z)|dz|$ the hyperbolic metric on Ω. Recall that

$$\lambda_D(z)|dz| = \frac{|dz|}{1-|z|^2}$$

is the hyperbolic metric on D, where D is the open unit disk in $\mathbf{C}$. The density λ_Ω of the hyperbolic metric on Ω is determined from

$$\lambda_\Omega(\varphi(z))|\varphi'(z)| = \lambda_D(z),$$

where $\varphi : D \to \Omega$ is any holomorphic universal covering projection of D onto Ω. A general discussion of the hyperbolic metric can be found in [1], [3], and [4].

A holomorphic function f on a hyperbolic region Ω is called a *Bloch function* if

$$\|f\|_B = \sup\left\{\frac{|f'(z)|}{\lambda_\Omega(z)} : z \in \Omega\right\} < \infty.$$

The quantity $\|f\|_B$ is called the Bloch norm of f. Let $\delta_\Omega(z) = dist\,(z, \partial\Omega)$; this is the radius of the largest disk in Ω with center z. We define the quasi-Bloch norm $\|f\|_{QB}$ by

$$\|f\|_{QB} = \sup\{\delta_\Omega(z)|f'(z)| : z \in \Omega\}.$$

Next, we define the Bloch number of a holomorphic function f in a hyperbolic region Ω. For more details, see [4] and [5]. For z in Ω let $r(z,f)$ be the radius of the largest unramified disk about $f(z)$

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in the Riemann image surface R_f of f; set $r(z,f)=0$ in case $f(z)$ is a branch point of R_f. An unramified disk in R_f with center $f(z)$ and radius r means an open disk $D(f(z),r)=\{w:|w-f(z)|<r\}$ with the property that there exists a simply connected region $\Delta\subset\Omega$ and $f|\Delta$ is a conformal mapping of Ω onto $D(f(z),r)$. By the *Bloch number* $r(f)$ of f is meant the radius of the largest unramified disk contained in $f(\Omega)$. That is,

$$r(f)=\sup\{r(z,f):z\in\Omega\}.$$

In this paper we show that for a function f holomorphic in a hyperbolic region, the quantities $\|f\|_B$, $\|f\|_{QB}$ and $r(f)$ are all comparable.

2. Main results

Lemma 1. *Let f be a nonconstant holomorphic function in a hyperbolic region Ω, and α be a complex number. Then for any complex number $z\in\Omega$*

$$(a)r(z,\alpha f)=|\alpha|\,r(z,f),(b)r(z,f-\alpha)=r(z,f).$$

Proof. To prove (a), let $r_1=r(z,\alpha f)$ and $r_2=r(z,f)$. Without loss of generality, we may assume that $\alpha\neq 0$. First, we show that $r_1\leq|\alpha|\,r_2$. If $r_1=0$, then we are done. Otherwise, there is a simply connected region $\Delta\subset\Omega$ such that $z\in\Delta,\alpha f|\Delta$ is univalent, and $(\alpha f)(\Delta)=D(\alpha f(z),r_1)$. Hence, $f|\Delta$ is univalent and

$$f(\Delta)=\frac{1}{\alpha}(\alpha f)(\Delta)=\frac{1}{\alpha}D(\alpha f(z),r_1)=D\left(f(z),\frac{r_1}{|\alpha|}\right).$$

Therefore, $\frac{r_1}{|\alpha|}\leq r_2$ or $r_1\leq|\alpha|\,r_2$. Next, we prove that $|\alpha|\,r_2\leq r_1$. Suppose $r_2\neq 0$. Then there exists a simply connected region $\Delta\subset\Omega$ such that $z\in\Delta$, $f|\Delta$ is univalent and $f(\Delta)=D(f(z),r_2)$. It follows that $\alpha f|\Delta$ is univalent and

$$(\alpha f)(\Delta)=\alpha D(f(z),r_2)=D(\alpha f(z),|\alpha|\,r_2).$$

This yields $|\alpha|\,r_2\leq r_1$. The proof of (b) is analogous.

Lemma 2. *Let f be a nonconstant holomorphic function in a hyperbolic region Ω, and let h be a conformal automorphism of Ω. Then for any $z \in \Omega$*

$$r(z, f \circ h) = r(h(z), f).$$

Proof. Let $r_1 = r(z, f \circ h)$ and $r_2 = r(h(z), f)$. Without loss of generality, we may assume that $r_1 \neq 0$ and $r_2 \neq 0$. First, we show that $r_1 \leq r_2$. There is a simply connected region $\Delta \subset \Omega$ such that $z \in \Delta$, $f \circ h|\Delta$ is univalent, and $(f \circ h)(\Delta) = D((f \circ h)(z), r_1)$. Then $\Delta^* = h(D) \subset \Omega$ is simply connected, $h(z) \in \Delta^*$, and $f|\Delta^*$ is univalent. Also, we have

$$f(\Delta^*) = (f \circ h)(\Delta) = D(f(h(z)), r_1).$$

Hence $r_1 \leq r_2$. Next, we prove that $r_2 \leq r_1$. There is a simply connected region $\Delta \subset \Omega$ such that $h(z) \in \Delta$, $f|\Delta$ is univalent, and $f(\Delta) = D(f(h(z)), r_2)$. Since $h \in Aut(\Omega)$, $\Delta^* = h^{-1}(\Delta) \subset \Omega$ is simply connected. Clearly, $z \in \Delta^*$, and $f \circ h|\Delta^*$ is univalent. We also have

$$(f \circ h)(\Delta^*) = f(\Delta) = D((f \circ h)(z), r_2).$$

This yields $r_2 \leq r_1$.

Lemma 3. *Let f be a nonconstant holomorphic function in a hyperbolic region Ω, and let $\varphi : D \longrightarrow \Omega$ be a holomorphic universal covering projection. Then for any $z \in D$*

$$r(z, f \circ \varphi) = r(\varphi(z), f).$$

Proof. Let $r_1 = r(z, f \circ \varphi)$ and $r_2 = r(\varphi(z), f)$. First, we show that $r_1 \leq r_2$. If $r_1 = 0$, then we are done. Otherwise, there is a simply connected region $\widetilde{\Delta} \subset D$ such that $z \in \widetilde{\Delta}$, $f \circ \varphi|\widetilde{\Delta}$ is univalent and $(f \circ \varphi)\left(\widetilde{\Delta}\right) = D((f \circ \varphi)(z), r_1)$. Then $\varphi|\widetilde{\Delta}$ is also univalent, $\Delta = \varphi\left(\widetilde{\Delta}\right)$ is simply connected, $f|\Delta$ is univalent and $f(\Delta) = D(f(\varphi(z)), r_1)$. This yields $r_1 \leq r_2$. Next, we prove that $r_2 \leq r_1$. There is a simply connected region $\Delta \subset \Omega$ such that $\varphi(z) \in \Delta$, $f|\Delta$ is univalent, and $f(\Delta) = D(f(\varphi(z)), r_2)$. Because Δ is simply connected and φ is a covering, there is a unique simply connected region $\widetilde{\Delta} \subset D$ such that $z \in \widetilde{\Delta}, \varphi\left(\widetilde{\Delta}\right) = \Delta$ and $\varphi|\widetilde{\Delta}$ is univalent. Then $f \circ \varphi|\widetilde{\Delta}$ is univalent and $(f \circ \varphi)\left(\widetilde{\Delta}\right) = D\left((f \circ \varphi)(z), r_2\right)$, so $r_2 \leq r_1$.

Let S be the family of all functions f holomorphic on the open unit disk D and normalized by $f'(0) = 1$. The *Bloch constant* β is the largest number such that any $f \in S$ has the property that $f(D)$ contains an unramified disk of radius β :

$$\beta = \inf\{r(f) : f \in S\}.$$

It is well known[2,p.47] that 0.433$< \frac{\sqrt{3}}{4} < \beta < 0.472$.

Theorem 1. *If f is holomorphic in a hyperbolic region Ω, then*

$$r(f) \leq \|f\|_B \leq \frac{r(f)}{\beta}.$$

Proof. To prove the left hand inequality, let $\varphi : D \to \Omega$ be a holomorphic universal covering projection. Then $f \circ \varphi$ is holomorphic in D, and

$$r(z, f \circ \varphi) \leq \frac{\left|(f \circ \varphi)'(z)\right|}{\lambda_D(z)}, \; z \in D.$$

This inequality is a result of Seidel and Walsh[6]. We have

$$\begin{aligned}\|f\|_B &= \sup\left\{\frac{|f'(w)|}{\lambda_\Omega(w)} : w \in \Omega\right\} \\ &= \sup\left\{\frac{\left|(f \circ \varphi)'(z)\right|}{\lambda_D(z)} : z \in D\right\} = \|f \circ \varphi\|_B .\end{aligned}$$

Therefore, Lemma 3 yields $r(f) \leq \|f\|_B$.

Now, we prove the right hand inequality. First, we assume the validity of the right hand inequality for the open unit disk D. Let $\varphi : D \to \Omega$ be a holomorphic universal covering projection. Then

$$\|f\circ\varphi\|_B \le \frac{r(f\circ\varphi)}{\beta}, \quad \|f\|_B = \|f\circ\varphi\|_B .$$

Therefore, Lemma 3 yields $\|f\|_B \le \frac{r(f)}{\beta}$. All that remains is to establish the right hand inequality in the special case $\Omega = D$. If f is constant on D, there is nothing to prove. Suppose f is not constant. Let a be a point in D such that $f'(a) \neq 0$. Then the function

$$h(z) = \frac{\lambda_D(a)}{|f'(a)|}\left[f\left(\frac{z+a}{1+\overline{a}z}\right) - f(a)\right]$$

is a nonconstant holomorphic function in D with $h'(0) = 1$. By Lemma 1, we have

$$r(z,h) = \frac{\lambda_D(a)}{|f'(a)|} r\left(z, f\left(\frac{z+a}{1+\overline{a}z}\right)\right).$$

Since the mapping $z \to \frac{z+a}{1+\overline{a}z}$ is a conformal automorphism of D, Lemma 2 yields

$$r\left(z, f\left(\frac{z+a}{1+\overline{a}z}\right)\right) = r\left(\frac{z+a}{1+\overline{a}z}, f\right) \le r(f).$$

Since $h \in S$, it follows that

$$\beta \le r(h) \le \frac{\lambda_D(a)}{|f'(a)|} r(f).$$

This completes the proof.

Koebe's one-quarter theorem[1,p.72] asserts the following: If f is univalent holomorphic in the open unit disk D and normalized by $f(0) = 0, |f'(0)| = 1$, then $f(z) \neq w_0$ for $|z| < 1$ implies $|w_0| \ge \frac{1}{4}$.

Theorem 2. *If f is holomorphic in a hyperbolic region Ω, then $r(f) \le 4\|f\|_{QB}$.*

Proof. Fix $a \in \Omega$ and set $b = f(a)$. Without loss of generality, we may assume that $r(a,f) \neq 0$. Then there exists a simply connected region $\Delta \subset \Omega$ such that $a \in \Delta$ and $f|\Delta$ is a conformal mapping of Δ onto $D(b, r(a,f))$. Set $g = (f|\Delta)^{-1}$. Define $h : D \to \Omega$ by

$$h(w) = \frac{g(b + r(a,f)w) - g(b)}{r(a,f)g'(b)}.$$

Then h is a one-to-one holomorphic in D and $h(0) = 0, h'(0) = 1$. Let z be a point in $\partial D(a, \delta_\Omega(a)) \cap \partial\Omega$. Then

$$\frac{z - g(b)}{r(a,f)g'(b)} \notin h(D).$$

The Koebe's $\frac{1}{4}$–theorem implies that $h(D) \supset D\left(0, \frac{1}{4}\right)$, so that

$$\left|\frac{z - g(b)}{r(a,f)g'(b)}\right| \geq \frac{1}{4}.$$

But $|z - g(b)| = |z - a| = \delta_\Omega(a)$ and $g'(b)f'(a) = 1$, so we have

$$\frac{\delta_\Omega |f'(a)|}{r(a,f)} \geq \frac{1}{4}.$$

This yields the desired inequality.

The inequality $\lambda_\Omega(z) \leq \frac{1}{\delta_\Omega(z)}$ is a direct consequence of the monotonicity theorem for the hyperbolic metric. This inequality gives $\|f\|_{QB} \leq \|f\|_B$. Therefore, we obtain the following result.

Corollary. . *If f is holomorphic in a hyperbolic region Ω, then*

$$\|f\|_{QB} \leq \|f\|_B \leq \frac{4}{\beta} \|f\|_{QB}.$$

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