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HYPERBOLIC CURVATURE ON PLANE REGIONS

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1. Introduction

Let γ be a smooth curve in a hyperbolic plane region Ω and $K_\Omega(z,\gamma)$ denote the hyperbolic curvature of the curve γ at the point $z \in \gamma$. Flinn and Osgood[5] proved that if f is a conformal mapping of a hyperbolic simply connected region Ω into a hyperbolic simply connected region Δ, then

$$\max\{K_\Omega(z,\gamma),2\} \leq \max\{K_\Delta(f(z), f\circ\gamma),2\}$$

for any smooth curve γ in Ω. This result gives the monotonicity theorem for the hyperbolic curvature.

Monotonicity Theorem. *Suppose Ω and Δ are hyperbolic simply connected regions in the complex plane $\mathbf{C}$ and $\Omega \subset \Delta$. If $K_\Omega(z,\gamma) \geq 2$, then for any smooth curve γ in Ω $K_\Omega(z,\gamma) \leq K_\Delta(z,\gamma)$.*

In this paper we investigate a type of monotonicity property for the hyperbolic curvature under a holomorphic mapping from a hyperbolic region to a hyperbolic region. In section 2 we obtain an inequality for the change in the euclidean curvature under a conformal mapping of the open unit disk into itself. In section 3 we discuss basic properties of the hyperbolic metric and hyperbolic curvature. In section 4 we prove that the monotonicity theorem for the hyperbolic curvature remains valid if Ω is a simply connected subregion of an arbitrary hyperbolic region Δ in the complex plane $\mathbf{C}$.

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2. The change of euclidean curvature

Suppose f is holomorphic and univalent in the open unit disk D and normalized by $f(0)=0, f'(0)=1$; say

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Then deBranges' Theorem[3] asserts that $|a_n| \leq n$ for $n=2,3,\ldots$ with equality if and only if $f=K_\theta$, where θ is a real number and

$$K_\theta(z)=\frac{z}{\left(1-e^{i\theta}z\right)^2}=z+2e^{i\theta}z^2+3e^{2i\theta}z^3+\ldots$$

is a Koebe function.

Lemma. *If f is a conformal mapping of D into itself with $f(0)=0$, then*

$$|f''(0)| \leq 4\,|f'(0)|\,(1-|f'(0)|)\,.$$

Proof. Since f is univalent, it follows from Schwarz' Lemma that $0<|f'(0)|\leq 1$. For a real number ϕ we consider

$$h(z)=\frac{1}{f'(0)}e^{-i\phi}K_\theta\left(f\left(e^{i\phi}z\right)\right).$$

Now,

$$K_\theta(f(z))=f(z)+2e^{i\theta}\left[f(z)\right]^2+\ldots$$

$$=f'(0)z+\left(\frac{f''(0)}{2}+2e^{i\theta}f'(0)^2\right)z^2+O\left(z^3\right)$$

so that

$$h(z)=z+\left[\frac{f''(0)}{2f'(0)}e^{i\phi}+2f'(0)e^{i(\theta+\phi)}\right]z^2+O(z^3).$$

The function h is holomorphic and univalent in D, and $h(0) = 0$, $h'(0) = 1$. Select θ and ϕ so that

$$\frac{f''(0)}{f'(0)}e^{i\phi} > 0 \text{ and } 2f'(0)e^{i(\theta+\phi)} > 0.$$

Then deBranges' Theorem gives

$$2 \geq \left| \frac{f''(0)}{2f'(0)}e^{i\phi} + 2f'(0)e^{i(\theta+\phi)} \right| = \frac{|f''(0)|}{2\,|f'(0)|} + 2\,|f'(0)|\,.$$

This completes the proof.

Let γ be a smooth curve in $\mathbf{C}$ with parametrization $z = z(t)$. The *euclidean curvature* $K_e\,(z,\gamma)$of the curve γ at the point $z = z(t)$ is the rate of change of the angle θ that the tangent vector makes with the positive real axis with respect to arc length:

$$K_e(z,\gamma) = \frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds} = \frac{1}{|z'(t)|}Im\left\{\frac{z''(t)}{z'(t)}\right\}.$$

If f is holomorphic and locally univalent in a neighborhood of γ, then $f \circ \gamma$ is also a smooth curve. The formula for the change of euclidean curvature under f is given by [7]

$$K_e\left(f(z), f \circ \gamma\right)|f'(z)| = K_e\left(z,\gamma\right) + Im\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

We now obtain an inequality for the change of euclidean curvature at the origin under a conformal mapping of the open unit disk into itself that fixes the origin.

Theorem 1. *Suppose f is a conformal mapping of D into itself with $f(0) = 0$. If γ is a smooth curve through the origin, then*

$$\max\left\{K_e(0, f \circ \gamma), 4\right\} \geq \max\left\{K_e(0,\gamma), 4\right\}.$$

Proof. We need only consider the case in which $K_e(0,\gamma) \geq 4$.The formula for the change of euclidean curvature gives

$$K_e(0, f\circ\gamma)\,|f'(0)| \geq K_e(0,\gamma) - \left|\frac{f''(0)}{f'(0)}\right|.$$

The previous Lemma gives $|f''(0)| \leq 4\,|f'(0)|\,(1-|f'(0)|)$.Therefore,

$$|f'(0)|\,[K_e(0, f\circ\gamma) - 4] \geq K_e(0,\gamma) - 4.$$

But $0 < |f'(0)| \leq 1$, so the desired result follows immediately.

3. The hyperbolic metric and hyperbolic curvature

We begin this section with a brief introduction to the hyperbolic metric. For a general discussion of the hyperbolic metric we refer the reader to [1], [6], and [9].

The hyperbolic metric on the open unit disk D in $\mathbf{C}$ is defined by

$$\lambda_D(z)\,|dz| = \frac{2\,|dz|}{1-|z|^2}.$$

A region Ω in $\mathbf{C}$ is called *hyperbolic* if the complement of Ω with respect to $\mathbf{C}$ contains at least two points. If a region Ω is hyperbolic, then, by the uniformization theorem [4,p.39], there is a holomorphic universal covering projection φ of D onto Ω. If Ω is simply connected, then φ is just a conformal mapping of D onto Ω. The density of the *hyperbolic metric* $\lambda_\Omega(z)\,|dz|$ on a hyperbolic region Ω is obtained from

$$\lambda_\Omega\,(\varphi(z))\,|\varphi'(z)| = \lambda_D(z),$$

where φ is any holomorphic universal covering projection of D onto Ω. The hyperbolic density is independent of the choice of the covering projection since

$$\frac{2\,|T'(z)|}{1-|T(z)|^2} = \frac{2}{1-|z|^2}$$

for any conformal automorphism T of D. The hyperbolic metric is invariant under holomorphic covering projections: If $f : \Omega \to \Delta$ is a holomorphic covering projection, then

$$\lambda_\Delta (f(z)) |f'(z)| |dz| = \lambda_\Omega (z) |dz| .$$

Example 1. *(1) For $a \in \mathbf{C}$ and $r > 0$ set $D(a,r) = \{z : |z-a| < r\}$ and*

$$\lambda_{D(a,r)}(z) = \frac{2r}{r^2 - |z-a|^2}.$$

Now, $f(z) = a + rz$ is a conformal mapping of D onto $D(a,r)$ and $\lambda_{D(a,r)}(f(z)) |f'(z)| = \lambda_D(z)$, so $\lambda_{D(a,r)}(z) |dz|$ is the hyperbolic metric on $D(a,r)$.

(2) Set $D' = \{z : 0 < |z| < 1\}$ and

$$\lambda_{D'}(z) = \frac{1}{|z| \log \frac{1}{|z|}}.$$

The holomorphic function $f(z) = \exp\left(\frac{z+1}{z-1}\right)$ maps D onto D' and $\lambda_{D'}(f(z)) |f'(z)| = \lambda_D(z)$, so $\lambda_{D'}(z) |dz|$ is the hyperbolic metric on D'.

(3) The hyperbolic metric on the upper half plane $H = \{z : Imz > 0\}$ is

$$\lambda_H(z) |dz| = \frac{1}{Imz}.$$

Next, we define the hyperbolic curvature of a smooth curve. We refer the reader to [5], [8], [9], [10] for further details. If γ is a smooth curve in a hyperbolic region Ω with parametrization $z = z(t)$, then the *hyperbolic curvature* of γ at the point $z = z(t)$ is given by

$$K_\Omega(z,\gamma) = \frac{1}{\lambda_\Omega(z)} \left[K_e(z,\gamma) - \frac{\partial \log \lambda_\Omega(z)}{\partial n} \right]$$

$$= \frac{1}{\lambda_\Omega(z)}\left[K_e(z,\gamma)+2Im\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right],$$

where is $n = n(z)$ the unit normal to γ at z.

Example 2. *For the open unit disk D we have*

$$K_D(z,\gamma) = \frac{1-|z|^2}{2}\left[K_e(z,\gamma)+2Im\left\{\frac{\overline{z}}{1-|z|^2}\frac{z'(t)}{|z'(t)|}\right\}\right]$$

$$= \frac{1}{2}\left(1-|z|^2\right)K_e(z,\gamma)+Im\left\{\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right\}.$$

In particular, $K_D(0,\gamma) = \frac{1}{2}K_e(0,\gamma)$. *Suppose* γ *is given by* $z(t) = a + re^{it}$, *where* a *is a real number,* $r > 0$ *and* $\alpha < t < \beta$ *is an interval so that* $z(t) \in D$ *for* $\alpha < t < \beta$. *Then*

$$K_D(z,\gamma) = \frac{1}{2}\left(1-|z|^2\right)K_e(z,\gamma)+Im\left\{\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right\}$$

$$= \frac{1}{2}\left(1-a^2-r^2-2ar\cos t\right)\frac{1}{r}+a\cos t+r = \frac{1+r^2-a^2}{2r}.$$

This gives the following results.

(1) If γ *is completely contained inside of D, then* $K_D(z,\gamma) > 1$.

(2) If γ *is tangent to the unit circle at 1, then* $K_D(z,\gamma) = 1$.

(3) If γ *is properly intersects the unit circle, then* $0 \le K_D(z,\gamma) < 1$.

(4) If γ *is orthogonal to the unit circle, then* $K_D(z,\gamma) = 0$. *In this case* γ *is a hyperbolic geodesic.*

Example 3. *For the hyperbolic region $D' = \{z : 0 < |z| < 1\}$ we have*

$$K_{D'}(z,\gamma) = \frac{1}{\lambda_{D'}(z)}\left[K_e(z,\gamma) + 2Im\left\{\frac{\partial \log \lambda_{D'}(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right]$$

$$= \frac{K_e(z,\gamma)}{\lambda_{D'}(z)} + \left(1 - \log\frac{1}{|z|}\right) Im\left\{\frac{\overline{z(t)}}{|z(t)|}\frac{z'(t)}{|z'(t)|}\right\}.$$

Suppose γ is given by $z(t) = \rho e^{it}$, where $0 \leq t \leq 2\pi$ and $0 < \rho < 1$.Then

$$K_{D'}(z,\gamma) = \frac{\frac{1}{\rho}}{\left(\rho \log\frac{1}{\rho}\right)^{-1}} + \left(1 - \log\frac{1}{\rho}\right).1 = 1.$$

This shows that the hyperbolic curvature of γ is independent of $\rho \in (0,1)$.

Example 4. *For the upper half plane H we have*

$$K_H(z,\gamma) = \frac{K_e(z,\gamma)}{\lambda_H(z)} + \frac{1}{\lambda_H(z)}2Im\left\{\frac{i}{2}\lambda_H(z)\frac{z'(t)}{|z'(t)|}\right\}$$

$$= \frac{K_e(z,\gamma)}{\lambda_H(z)} + Re\left\{\frac{z'(t)}{|z'(t)|}\right\}.$$

Suppose γ is the line $z(t) = x_0 + te^{i\theta}, t > 0$. Then

$$K_H(z,\gamma) = 0 + Ree^{i\theta} = \cos\theta.$$

For $\theta = \frac{\pi}{2}, \gamma$ is the hyperbolic geodesic and $K_H(z,\gamma) = 0$.

Now, we show that the hyperbolic curvature is invariant under holomorphic covering projections.

Theorem 2. *Suppose Ω and Δ are hyperbolic regions in $\mathbf{C}$ and $f : \Omega \to \Delta$ is a holomorphic covering projection. Then $K_\Omega(z,\gamma) = K_\Delta(f(z), f \circ \gamma)$ for any smooth curve γ in Ω.*

Proof. Let $w = f(z)$ and $\delta = f \circ \gamma$. From $\lambda_\Omega(z) = \lambda_\Delta(f(z))\,|f'(z)|$, we obtain

$$\frac{\partial \log \lambda_\Omega(z)}{\partial z} = \frac{\partial}{\partial z}\left[\log \lambda_\Delta(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log \overline{f'(z)}\right]$$

$$= \frac{\partial \log \lambda_\Delta(w)}{\partial w} f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}.$$

So, by the transformation law for euclidean curvature, we have

(*)

$$K_\Omega(z,\gamma) = \frac{K_e(z,\gamma)}{\lambda_\Omega(z)} + \frac{1}{\lambda_\Omega(z)} 2Im\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}$$

$$= \frac{1}{\lambda_\Delta(w)\,|f'(z)|}\left[K_e(w,\delta)\,|f'(z)| - Im\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}\right]$$

$$+ \frac{1}{\lambda_\Delta(w)\,|f'(z)|} 2Im\left[\frac{\partial \log \lambda_\Delta(w)}{\partial w} f'(z)\frac{z'(t)}{|z'(t)|} + \frac{1}{2}\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right]$$

$$= \frac{1}{\lambda_\Delta(w)}\left[K_e(w,\delta) + 2Im\left\{\frac{\partial \log \lambda_\Delta(w)}{\partial w}\frac{f'(z)}{|f'(z)|}\frac{z'(t)}{|z'(t)|}\right\}\right].$$

But $\frac{w'(t)}{|w'(t)|} = \frac{f'(z)}{|f'(z)|}\frac{z'(t)}{|z'(t)|}$, so the desired result follows from (*).

4. The change of hyperbolic curvature

In the monotonicity theorem for the hyperbolic curvature, Ω is a simply connected subregion of the hyperbolic simply connected region Δ. Does the monotonicity theorem extend to multiply connected regions? The following example shows that in general the result fails if Δ is simply connected while Ω is allowed to be non-simply connected.

Example 5. *Consider* $\Omega = D' = D - \{0\}$*and* $\Delta = D$. *Let*

$$\gamma : z(t) = x_0 - it, |t| < \sqrt{1 - x_0^2}, 0 < x_0 < 1.$$

Now, $K_e(z, \gamma) = 0$ *so that*

$$K_D(z,\gamma) = Im\left\{\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right\},$$

$$K_{D'}(z,\gamma) = \left(1 - \log\frac{1}{|z|}\right) Im\left\{\frac{\overline{z(t)}z'(t)}{|z(t)|\,|z'(t)|}\right\}.$$

We have $\frac{z'(t)}{|z'(t)|} = -i$ *so that*

$$K_D(x_0,\gamma) = Im\{-ix_0\} = -x_0 < 0$$

and

$$K_{D'}(x_0,\gamma) = \left(1 - \log\frac{1}{x_0}\right) Im\{-i\} = \log\frac{1}{x_0} - 1 \in (0,\infty)$$
$$\text{if } x_0 \in \left(0, \frac{1}{e}\right).$$

In particular, for $0 < x_0 < \frac{1}{e^3}$ *we have*

$$K_{D'}(x_0,\gamma) > 2 \text{ while } K_D(x_0,\gamma) = -x_0 < 0.$$

Now, we prove that the monotonicity theorem for the hyperbolic curvature remains valid if Ω is simply connected subregion of an arbitrary hyperbolic region Δ.

Theorem 3. *Suppose Δ is a hyperbolic region in the complex plane* **C** *and Ω is a simply connected subregion of Δ. If γ is a smooth curve in Ω, then*

$$\max\{K_\Omega(z,\gamma), 2\} \le \max\{K_\Delta(z,\gamma), 2\}.$$

Proof. Fix $a \in \gamma$. We need only consider the case in which $K_\Omega(a,\gamma) \geq 2$. Let $f : D \to \Omega$ be a conformal mapping with $f(0) = a$ and $h : D \to \Delta$ be a holomorphic universal covering projection with $h(0) = a$. Since Ω is simply connected and h is a covering projection, the Monodromy Theorem[2, p.295] implies that the branch of h^{-1} that satisfies $h^{-1}(a) = 0$ is holomorphic and single-valued in Ω. Thus, $h^{-1} : \Omega \to D$ maps Ω into D. Since $h \circ h^{-1}$ is the identity mapping on Ω, h^{-1} is actually univalent on Ω. Let $g : \Omega \to \Delta$ be the inclusion map. Define $\widetilde{g} = h^{-1} \circ g \circ f$. Then $\widetilde{g}$ is holomorphic in D, univalent in D and $\widetilde{g}(0) = 0$. Let $\delta = g \circ \gamma$. If $\widetilde{\gamma} = f^{-1} \circ \gamma$ and $\widetilde{\delta} = h^{-1} \circ \delta$, then, by Theorem 2, we have

$$K_D(0,\widetilde{\gamma}) = K_\Omega(a,\gamma),\ K_D\left(0,\widetilde{\delta}\right) = K_\Delta(a,\delta).$$

So it suffices to show that $K_D(0,\widetilde{\gamma}) \leq K_D\left(0,\widetilde{\delta}\right)$. This is equivalent to

$$(**) \qquad K_e(0,\widetilde{\gamma}) \leq K_e\left(0,\widetilde{\delta}\right)$$

We note that

$$\widetilde{\delta} = h^{-1} \circ \delta = h^{-1} \circ g \circ \gamma = h^{-1} \circ g \circ f \circ \widetilde{\gamma} = \widetilde{g} \circ \widetilde{\gamma}.$$

Because

$$K_e(0,\widetilde{\gamma}) = 2K_D(0,\widetilde{\gamma}) = 2K_\Omega(a,\gamma) \geq 4,$$

the inequality (2) follows from Theorem 1.

Corollary 1. *Suppose Ω and Δ are hyperbolic regions in* **C** *with Ω simply connected. If $f : \Omega \to \Delta$ is a conformal mapping, then for any smooth curve γ in Ω*

$$\max\{K_\Omega(z,\gamma), 2\} \leq \max\{K_\Delta(f(z), f \circ \gamma), 2\}.$$

Proof. Since the hyperbolic curvature is a conformal invariant, we have

$$K_\Omega(z,\gamma) = K_{f(\Omega)}\left(f(z), f\circ\gamma\right).$$

Theorem 3 yields

$$\max\left\{K_{f(\Omega)}\left(f(z), f\circ\gamma\right), 2\right\} \le \max\left\{K_\Delta\left(f(z), f\circ\gamma\right), 2\right\},$$

so this establishes the Corollary 1.

Corollary 2. *Suppose Ω is any hyperbolic region in* **C**, *γ is a smooth curve in Ω, $a \in \gamma$ and $\delta_\Omega(a) = dist(a, \partial\Omega)$. If $K_e(a,\gamma) \ge \frac{4}{\delta_\Omega(a)}$, then $K_\Omega(a,\gamma) \ge 2$.*

Proof. Consider the disk $D(a,\delta) \subset \Omega$, where $\delta = \delta_\Omega(a)$. Then

$$K_{D(a,\delta)}(a,\gamma) = \frac{\delta}{2}K_e(a,\gamma)$$

yields $K_{D(a,\delta)}(a,\gamma) \ge 2$. Because $D(a,\delta)$ is simply connected, Theorem 3 gives

$$K_\Omega(a,\gamma) \ge K_{D(a,\delta)}(a,\gamma) \ge 2.$$

References

1 L V Ahlfors, *Conformal invariants Topics in geometric function theory*, McGraw-Hill, New York, 1973.
2 L V. Ahlfors, *Complex Analysis, 3-rd ed*, McGraw-Hill, New York, 1979
3. L deBranges, *A proof of the Bieberbach conjecture*, Acta Math **154** (1985), 137–152
4. S D Fisher, *Function theory on planar domains A second course in complex analysis*, John Wiley & Sons, New York, 1983
5. B Flinn and B Osgood, *Hyperbolic curvature and conformal mapping*, Bull. London Math Soc **18** (1986), 272–276
6 G.M Goluzin, *Geometric theory of functions of a complex variable. Translations of Mathematical Monographs*, **26**, Amer. Math. Soc., Providence, 1969.
7. A.W. Goodman, *Curvature under an analytic transformation*, J London Math. Soc **43** (1968), 527–533.
8. D. Minda, *Applications of hyperbolic convexity to euclidean and spherical convexity*, J Analyse Math. **49** (1987), 90–105

9. D. Minda, *Hyperbolic curvature on Riemann surfaces*, Complex Variables Theory Appl. **12** (1989), 1–8.
9. B. Osgood, *Some properties of $\frac{f''}{f'}$ and the Poincaré metric*, Indiana Univ Math J. **31** (1982), 449–461.

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