NOTES ON PRIMARY IDEALS

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We know that a primary ideal of a commutative ring $R$ is defined to be an ideal of $R$ such that if $xy \in I$ and $x \notin I$, then $y^n \in I$ for some positive integer $n$. B.S. Chew and J. Neggers extended the concept to general rings in their paper [1].

In this paper we will give slightly different definitions of the strongly primary ideal of B.S. Chew and J. Neggers. We will call this $w$-strongly primary ideal. We will show that every $w$-strongly primary ideal is primary ideal in a commutative ring and a matrix ring of $w$-strongly primary ring is also $w$-strongly primary ring. Through this paper we assume that $R$ is a ring with identity and every $R$-module $M$ is unitary left $R$-module.

We recall the definitions of primary and strongly primary ideals of B.S. Chew and J. Neggers.

DEFINITION [1]. Suppose $R$ is a ring. An ideal $I$ of $R$ is called left primary if there is a faithful indecomposable $R/I$-module $M$. Moreover if $M$ is both Artinian and noetherian $R/I$-module, then $I$ is called left strongly primary.

It is known that every strongly primary ideal is a primary ideal in usual sense in a commutative ring and every primary ideal is a left primary ideal[1]. Usually we call a ring $R$ left primary and left strongly primary if 0 is left primary and strongly primary ideal respectively. Since the integer ring $\mathbb{Z}$ has no faithful noetherian and artinian $\mathbb{Z}$-module, $\mathbb{Z}$ is not strongly primary. Thus we know that primeness does not imply strongly primariness. Either strongly primariness does not imply primeness because $9\mathbb{Z}$ is a strongly primary ideal of an integer ring $\mathbb{Z}$ but not prime.

But we have the following propositions easily.

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PROPOSITION 1. Let $R$ be a commutative principal ideal domain. Then every nontrivial prime ideal is a strongly primary ideal.

*Proof.* Since $R$ is a commutative principal domain, every nontrivial principal prime ideal $I$ is maximal. So $R/I$ is a field and clearly strongly primary.

PROPOSITION 2. If $R$ is a semisimple primary ring, then $R$ is strongly primary (in fact $R$ is primitive).

*Proof.* Let $M$ be a faithful indecomposable $R$-module. Since $R$ is semisimple, $M$ is semisimple. So $M$ is simple because $M$ is indecomposable. Thus $R$ has a faithful indecomposable artinian and noetherian.

PROPOSITION 3. If a left artinian ring $R$ has no nontrivial idempotents, then $R$ is strongly primary.

*Proof.* Let $M = _R R$. Then $\text{End}_R(M) \cong R$. Since $R$ has no nontrivial idempotents, $M$ is indecomposable and $M = R$ is artinian and noetherian $R$-module.

PROPOSITION 4. If $R$ is semisimple, the intersection of strongly primary ideals is zero.

*Proof.* Let $R = \bigoplus_{i \in I} I_i$, where $I_i$ is minimal left ideal and $J_i = \text{ann}_R(I_i) = \{r \in R \mid rI_i = 0\}$. Clearly $J_i$ is two sided ideal and strongly primary for $I_i$ is a faithful indecomposable artinian and noetherian $R/J_i$-module. Clearly $\bigcap_{i \in I} J_i = \{0\}$.

Also we know that if $R$ is a right Goldie ring, then the intersection of all primary ideals is zero by similar method.

The following theorem shows that if $R$ is a left artinian primary ring and $R$ have an injective left nonzero ideal, then $R$ is a left strongly primary ring.

THEOREM 1. Let $R$ be a left artinian and $R$ have an injective left nonzero ideal. Then if $R$ is a left primary ring, $R$ is a left strongly primary ring.

*Proof.* Suppose $L$ is an injective left ideal. Then $L$ is a direct summand of $R$, that is $R = L \oplus L'$ for a suitable left ideal $L'$ of $R$. Since $R$ is left artinian, we can refine this decomposition into an indecomposable
direct decomposition of $R$. Let $R = I \oplus I'$ where a left ideal $I$ is a direct summand of $L$ (so $I$ is injective) and $I$ is indecomposable left $R$-module. Since $I$ is left artinian, $I$ contains a simple left ideal $J$. Then $I$ is the injective envelope of $J$ for $I$ is indecomposable and injective. Thus $J$ is a unique simple left ideal of $I$. Since $R$ is primary, $R$ has a faithful indecomposable $R$-module $M$. Then there exists an element $m$ in $M$ such that $Jm \neq 0$ for $JM \neq 0$. We can define an $R$-module homomorphism $\Phi_m$ from $I$ into $M$ as follows $\Phi_m(a) = am$. Then $\ker \Phi_m = \{a \mid am = 0\}$ does not contain $J$. So $\ker \Phi_m = \{0\}$ for $J$ is the unique minimal left ideal of $I$. Thus $\Phi_m$ is a monomorphism and $Im \cong I$ is an injective submodule of $M$. Moreover $Im$ is a direct summand of $M$ by injectiveness of $Im$. Clearly $Im \cong M$. Thus $R$ has a faithful indecomposable artinian and noetherian $R$-module $M$ for $I$ is left artinian and noetherian.

We define $w$-strongly primary ideal as following.

**DEFINITION.** An ideal $I$ of a ring $R$ is called $w$-strongly primary ideal if there exists a faithful $R/I$-module $M$ such that $\text{End}_{R/I}(M)$ is local ring and its Jacobson radical is nil ideal.

We know that if $M$ is indecomposable artinian and noetherian $R$-module, $\text{End}_R(M)$ is local ring and its Jacobson radical is nilpotent[2]. Thus every strongly primary ring is $w$-strongly primary.

The following theorems show that every commutative $w$-strongly primary ring is primary and a matrix ring of $w$-strongly primary ring is also $w$-strongly primary ring.

**THEOREM 2.** Let $R$ be a commutative ring. If $R$ is $w$-strongly primary, $R$ is primary.

**Proof.** Let $M$ be a faithful $R$-module and $S = \text{End}_R(M)$ be local and its Jacobson radical be nil. We imbeds $R$ in $S$ via $T_a(m) = am$ (in fact $a$ is mapped into $T_a$). Let $ab = 0$ and $b \neq 0$ in $R$. Then $T_aT_b = T_{ab} = 0$ and $T_b \neq 0$. So $T_a \in \text{rad}(S)$. Since $\text{rad}(S)$ is nil, $(T_a)^n = 0$ for some $n$. Thus $(T_a)^n = 0$ implies $a^nM = 0$ for $(T_a)^n = T_a^n$. Since $M$ is a faithful $R$-module, $a^n = 0$.

**THEOREM 3.** $R$ is a $w$-strongly primary ring iff $M_n(R)$ is a $w$-strongly primary ring where $M_n(R)$ is $(n, n)$ matrix ring over $R$. 
Proof. If $M$ is a faithful $R$-module such that $\text{End}_R(M)$ is local and its Jacobson radical is nil. Let $N = M \oplus \cdots \oplus M$ ($n$-copies) as a direct sum of groups. We define $M_n(R)$-action as following:

$$(\tau_{ij})(m_1, \ldots, m_i, \ldots, m_n) \overset{\text{def}}{=} (\ldots, \sum_{j=1}^{n} \tau_{ij} m_j, \ldots)$$

Then $N$ is a faithful $M_n(R)$-module. We will prove that $\text{End}_R(M) \cong \text{End}_{M_n(R)}(N)$. At first we can define a ring homomorphism $\Psi$ from $\text{End}_R(M)$ into $\text{End}_{M_n(R)}(N)$ as following:

$$\Psi(\sigma)(m_1, \ldots, m_i, \ldots, m_n) \overset{\text{def}}{=} (\sigma(m_1), \ldots, \sigma(m_i), \ldots, \sigma(m_n))$$

for every $\sigma \in \text{End}_R(M)$. By simple calculation, we know that $\Psi(\sigma)$ is an element of $\text{End}_{M_n(R)}(N)$ and $\Psi$ is a ring homomorphism. On the other hand if $\tau$ is any $M_n(R)$-module homomorphism of $N$.

Since

$$\tau(0, \ldots, m_i, 0, \ldots, 0) = \tau(E_{ii}(0, \ldots, m_i, 0, \ldots, 0)$$

$$= E_{ii} \tau(0, \ldots, m_i, 0, \ldots, 0)$$

$$= E_{ii}(m'_1, \ldots, m'_i, \ldots, m'_n)$$

$$= (0, \ldots, 0, m'_i, 0, \ldots, 0),$$

we have $\tau(0, \ldots, m_i, 0, \ldots, 0) = (0, \ldots, m'_i, 0, \ldots, 0)$ where $E_{ij}$ is the matrix whose element of $i$-th row and $j$-th column is 1 and otherwise is 0. For each $i$, we can define $\sigma_i$ as $\sigma_i(m) = \pi_i \tau_i(m)$ where $\pi_i$ is $i$-th injection from $M$ into $N$ and $\tau_i$ is $i$-th projection from $N$ into $M$. Then clearly $\sigma_i$ is $R$-module homomorphism of $M$.

Since

$$\sigma_i(m) = \pi_i \tau_i(m)$$

$$= \pi_i \tau(E_{ij} \iota_j(m))$$

$$= \pi_i E_{ij} \tau_j(m)$$

$$= \pi_i E_{ij}(0, \ldots, \sigma_j(m), \ldots, 0, \ldots, 0)$$

$$= \sigma_j(m),$$
we have $\sigma_i = \sigma_j = \sigma$ for every $i \neq j$. Thus $\tau = \Psi(\sigma)$. It is clear that $\Psi$ is one to one. Hence $\Psi$ is an isomorphism and $\text{End}_R(M) \cong \text{End}_{M_n(R)}(N)$.

Conversely $N$ is a faithful $M_n(R)$-module. Define $N_i = E_{ii}N$. Then $N = N_1 \oplus \ldots \oplus N_n$ as a direct sum of abelian groups and $N_i \cong N_j$ for $i \neq j$. Each $N_i$ is an $R$-module via $rn = rE_{ii}n$ for $n \in N_i$. Clearly $M = N_i$ is a faithful $R$-module and $\text{End}_R(M) \cong \text{End}_{M_n(R)}(N)$. Thus theorem is proved.

References

1. B.S. Chew and J. Neggers, Primary Ideals, Korean Math Soc. (2)20 (1984), 141–146

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