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$R[X]$–LINEAR MAPS OF THE MACAULAY–NORTHCOTT MODULE

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1. Introduction

Northcott in [3] considered the module $K[x^{-1}]$ of inverse polynomials over the polynomial ring $K[x]$ (with K a field). The idea for this module came from Macaulay's work in [1]. McKerrow in [2] generalized Northcott's work and considered the module $M[x^{-1}]$ over $R[x]$ (with R a ring and M a left R-module). If E is an injective left R-module and R is left noetherian then $E[x^{-1}]$ is an injective left $R[x]$-module (see [2]). In [4] and [5] we studied the behaviors of these so-called Macaulay-Northcott modules when we apply the torsion and extension functors to them. In this paper we will consider the $R[x]$-linear maps of these modules.

DEFINITION 1.1. Let R be a ring and M be a left R-module then $M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1}.$$

We call $M[x^{-1}]$ a Macaulay-Northcott Module.

DEFINITION 1.2. Let $\mathcal{C}$ be the category of left R-module and $\mathcal{D}$ be the category of left $R[x]$-module. Let $f :_R M \to_R N$ be a linear map, then $T : \mathcal{C} \to \mathcal{D}$ defined by $T(M) = M[x^{-1}]$ and $T(f) = f$ (where $f(m_0+m_1x^{-1}+\cdots+m_nx^{-n}) = f(m_0)+f(m_1)x^{-1}+\cdots+f(m_n)x^{-n}$) is a functor between $\mathcal{C}$ and $\mathcal{D}$. We call T the Macaulay-Northcott Functor.

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THEOREM 1.3. *There is a natural isomorphism*

$$Hom_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong Hom_R(M,N)[[x]].$$

Proof. See Theorem 4.1 [4].

Suppose $\phi : M \to N$ is a R-linear map, then we have the obvious map $M[x^{-1}] \to N[x^{-1}]$, namely $\phi + 0 \cdot x + 0 \cdot x^2 + \cdots \in \mathrm{Hom}_R(M,N)[[x]]$.

2. The Macaulay-Northcott Module

PROPOSITION 2.1. *Let M be an essential extension of N as a left R-module then $M[x^{-1}]$ is an essential extension of $N[x^{-1}]$.*

Proof. Let $m_0 + m_1 x^{-1} + \cdots + m_i x^{-i} \in M[x^{-1}]$ w.l.o.g. let $m_i \neq 0$ then there is $r_i \in R$, $r_i \neq 0$ such that $m_i r_i \in N$, $m_i r_i \neq 0$. So $r_i x_i \, (m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = r_i m_i \in N[x^{-1}]$. Hence $M[x^{-1}]$ is an essential extension of $N[x^{-1}]$.

REMARK 2.2. Let R be left noetherian. If E is an injective envelope of M then $E[x^{-1}]$ is an injective envelope of $M[x^{-1}]$.

Note that if ${}_R M \subset_R N$, then

$$\frac{N[x^{-1}]}{M[x^{-1}]} \cong \frac{N}{M}[x^{-1}].$$

PROPOSITION 2.3. *If $0 \to M \to E^0 \to E^1 \to \cdots$ is a minimal injective resolution of M as a left R-module then*

$$0 \to M[x^{-1}] \to E^0[x^{-1}] \to E^1[x^{-1}] \to \cdots$$

is a minimal injective resolution.

Proof. Let $0 \to M \xrightarrow{\epsilon} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} \cdots$ and $0 \to M[x^{-1}] \xrightarrow{\bar{\epsilon}} E^0[x^{-1}] \xrightarrow{\bar{d_0}} E_1[x^{-1}] \xrightarrow{\bar{d_1}} \cdots$. Let $m_0 + m_1 x^{-1} + \cdots + m_i x^{-i} \in M[x^{-1}]$. Then $\bar{d_0} \circ \bar{\epsilon}(m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = (d_0 + 0 \cdot x + 0 \cdot x^2 + \cdots) \circ (\epsilon + 0 \cdot x + 0 \cdot x^2 + \cdots)(m_0 + \cdots + m_i x^{-i}) = d_0(\epsilon(m_0)) + d_0(\epsilon(m_1))x^{-1} + \cdots + d_0(\epsilon(m_i))x^{-i} = 0$ So im $(\bar{\epsilon}) \subset$ ker $(\bar{d_0})$. Let $e_0 + e_1 x^{-1} + \cdots +$

$e_i x^{-i} \in \ker(\bar{d_0})$. Then $d(e_0) + d(e_1)x^{-1} + \cdots + d(e_i)x^{-i} = 0$. So $d(e_0) = d(e_1) = \cdots = d(e_i) = 0$. So $e_0, e_1, ..., e_i \in \mathrm{im}(\epsilon)$. So there exist $m_0, m_1, ..., m_i$ such that $\epsilon(m_0) = e_0, \epsilon(m_1) = e_1, ..., \epsilon(m_i) = e_i$. Now $\bar{\epsilon}(m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = e_0 + e_1 x^{-1} + \cdots + e_i x^{-i}$. So $\mathrm{im}(\bar{\epsilon}) = \ker(\bar{d_0})$. By the same process we have $\mathrm{im}(\bar{d_k}) = \ker(\bar{d}_{k+1})$. And by Remark 2.2, $E^{k+1}[x^{-1}]$ is an injective envelope of $E^k[x^{-1}]$. So $0 \to M[x^{-1}] \to E^0[x^{-1}] \to \cdots$ is a minimal injective resolution of $M[x^{-1}]$ as a left $R[x]$-module.

PROPOSITION 2.4. *Let $\phi : M[x^{-1}]$ be $R[x]$-linear map. Then $\phi(M) \subset M$.*

Proof.. Suppose $m \in M$ and $\phi(m) = f \notin M$ and $f = m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}$. Then for $x \in R[x]$, $\phi(xm) = 0$ and $x\phi(m) = m_1 + m_2 x^{-1} + \cdots + m_n x^{-n} \neq 0$ So $x\phi(m) \neq \phi(xm)$. This contradicts the fact that ϕ is a $R[x]$-linear map. So $\phi(M) \subset M$.

PROPOSITION 2.5. *Let $\phi : M[x^{-1}] \to M[x^{-1}]$ be $R[x]$-linear map.*

1) If $M \overset{_M|\phi|_M}{\to} M$ is one to one so is ϕ.

2) If $M \overset{_M|\phi|_M}{\to} M$ is an isomorphism so is ϕ.

Proof of 1). Suppose ϕ is not one to one. Let $h = \ker(\phi)$ for $h = m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}$ and w.l.o.g. $m_n \neq 0$. Then $\phi(h) = \phi(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = 0$. Since ϕ is $R[x]$-linear map, $x\phi(m_0) + x\phi(m_1 x^{-1}) + \cdots + x\phi(m_n x^{-n}) = \phi(m_1) + \phi(m_2 x^{-1}) + \cdots + \phi(m_n x^{-n+1}) = 0$. Multiply x on the left hand side again, then we have $\phi(m_2) + \phi(m_3 x^{-1}) + \cdots + \phi(m_n x^{-n+2}) = 0$. Repeat this process until we have $\phi(m_n) = 0$. So $m_n \in \ker({}_M|\phi|_M)$. This contradicts the fact that ${}_M|\phi|_M$ is 1-1. So ϕ is 1-1.

Proof of 2). Suppose ${}_M|\phi|_M$ is an isomorphism. Then by 1) ϕ is one to one so we want to show ϕ is onto. Let $f \in M[x^{-1}]$ and $f = m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}$. Suppose $\phi(g) = f$ for $g \in M[x^{-i}]$. Then $x^i\phi(g) = m_i$. So let $g = n_0 + n_1 x^{-1} + \cdots + n_i x^{-i}$. Then $x^i g = n_i$. Now choose n_i such that $\phi(n_i) = m_i$. Let $\phi(n_i x^{-1}) = c_{i-1} + m_i x^{-1}$. Choose n_{i-1} such that $\phi(n_{i-1}) = m_{i-1} - c_{i-1}$. And let $\phi(n_{i-1}x^{-1}) + \phi(n_i x^{-2}) = c_{i-2} + m_{i-1}x^{-1} + m_i x^{-2}$. Choose n_{i-2} such that $\phi(n_{i-2}) = n_{i-2} - c_{i-2}$. By this process we can get $n_{i-3}, ..., n_0$ and we have $\phi(g) = f$. So ϕ is onto. So ϕ is an assumption.

PROPOSITION 2.6. *$\sigma : M[[x^{-1}]]/M[x^{-1}] \to M[[x^{-1}]]/M[x^{-1}]$ by $f + M[x^{-1}] \to x(f + M[x^{-1}])$ is an isomorphism.*

Proof. Let $f + M[x^{-1}] \in \ker(\sigma)$ and let $f = a_0 + a_1x^{-1} + a_2x^{-2} + \cdots$. Then $\sigma(f + [Mx^{-1}]) = x(f + M[x^{-1}]) = M[x^{-1}]$. So $f + M[x^{-1}] = M[x^{-1}]$. So f is one to one. Let $f + M[x^{-1}] = (a_0 + a_1x^{-1} + a_2x^{-2} + \cdots) + M[x^{-1}] \in M[[x^{-1}]]/M[x^{-1}]$. Let $g + M[x^{-1}] = (a_0x^{-1} + a_2x^{-2} + a_2x^{-3} + \cdots) + M[x^{-1}]$, then $\sigma(g + M[x^{-1}]) = f + M[x^{-1}]$. Hence σ is onto. So σ is an isomorphism.

THEOREM 2.7. *Let $\phi : E[x^{-1}] \to E[x^{-1}]$ be a linear map for ${}_RE$ injective, then there is a $\psi : E[[x^{-1}]] \to E[[x^{-1}]]$ such that ${}_{E[x^{-1}]}|\psi|_{E[x^{-1}]} = \phi$. Moreover ψ is not unique in general.*

Proof. Since $E[[x^{-1}]]$ is an injective left $R[x]$-module we can complete the following diagram

$$\begin{array}{ccc} E[x^{-1}] & \hookrightarrow & E[[x^{-1}]] \\ \phi \downarrow & \swarrow \psi & \\ E[[x^{-1}]] & & \end{array}$$

So we have ψ such that ${}_{E[x^{-1}]}|\psi|_{E[x^{-1}]} = \phi$.

Let $\psi_1, \psi_2 : E[[x^{-1}]] \to E[[x^{-1}]]$ and ${}_{E[x^{-1}]}|\psi_i|_{E[x^{-1}]} = \phi$ for $i = 1, 2$. Then $\psi_1|_{E[x^{-1}]} = \phi$, $\psi_2|_{E[x^{-1}]} = \phi$ and $\psi_1 - \psi_2|_{E[x^{-1}]} = 0$. So $E[x^{-1}] \subset \ker(\psi_1 - \psi_2)$. So we have an induced map

$$E[[x^{-1}]]/E[x^{-1}] \to E[[x^{-1}]] \text{ by } f + E[x^{-1}] = (\psi_1 - \psi_2)(f).$$

Now consider the following. Let $\phi : E[x^{-1}] \to E[x^{-1}]$ be a linear map. Let $\psi_1 : E[[x^{-1}]] \to E[[x^{-1}]]$ such that ${}_{E[x^{-1}]}|\psi_1|_{E[x^{-1}]} = \phi$. Let $\sigma : E[[x^{-1}]]/E[x^{-1}] \to E[[x^{-1}]]$ be a non zero linear map, then there is a non zero linear map $\tau : E[[x^{-1}]] \to E[[x^{-1}]]$ such that $\tau(f) = \sigma(f + E[[x^{-1}]])$ and $E[x^{-1}] \subset \ker(\tau)$. Let $\psi_2 : E[[x^{-1}]] \to E[[x^{-1}]]$ such that $\psi_2 = \psi_1 - \tau$, then

$${}_{E[x^{-1}]}|\psi_2|_{E[x^{-1}]} = {}_{E[x^{-1}]}|\psi_1 - \tau|_{E[x^{-1}]} = {}_{E[x^{-1}]}|\psi_1|_{E[x^{-1}]} = \phi.$$

So there is ψ_2 such that $\psi_1 \neq \psi_2$.

EXAMPLE 2.8. Let $R = Z$ and $E = Q$. Let $\phi : Q[x^{-1}] \to Q[x^{-1}]$ be a linear map. Let $\psi_1 : Q[[x^{-1}]] \to Q[[x^{-1}]]$ be a linear map such that ${}_{Q[x^{-1}]}|\psi_1|_{Q[x^{-1}]} = \phi$. Consider $Q[[x^{-1}]]/Q[x^{-1}]$ and $Q[[x^{-1}]]$ as left $Z[x]$-modules. Let $f = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots \in Q[[x^1]]$. For any non zero $g \in Z[x]$ we claim that $g \cdot f \notin Q[x]$. Suppose $g \cdot f = h \in Q[x]$ and w.l.o.g. $\deg h = n$. Then $h^{(n+1)}(x) = 0$, but $(g \cdot f)^{(n+1)}(x) \neq 0$ become $(g \cdot f)^{(n+1)}(x)$ has always $g(x)e^x$ term and the degrees of the rest terms are strictly less than the degree of $g(x)$. So $g \cdot f \notin Q[x]$. Let $\bar{f} = 1 + x^{-1} + \frac{x^{-2}}{2!} + \frac{x^{-3}}{3!} + \cdots \in Q[[x^{-1}]]$. For non zero $g \in Z[x]$ we claim $g \cdot \bar{f} \notin Q[x^{-1}]$. Let $g = a_0 + a_1 x + a_2 x^2 + \cdots + a_{i-1}x^{i-1} + a_i x^i$. Then $(a_i + a_{i-1}x^{-1} + \cdots + a_0 x^{-i}) \cdot \bar{f}$ and $(a_i + a_{i-1}x + \cdots + a_0 x^i) \cdot f$ have some coefficient for each x^{-n} and x^n terms. So $(a_i + a_{i-1}x^{-1} + \cdots + a_0 x^{-i}) \cdot \bar{f} \notin Q[x^{-1}]$. So $x^i[(a_i + a_{i-1}x^{-1} + \cdots + a_0 x^{-i}) \cdot \bar{f}] \notin Q[x^{-1}]$. So $g \cdot \bar{f} \notin Q[x^{-1}]$. So there is a linear map $\sigma :$ $[[x^{-1}]]/Q[x^{-1}] \to Q[[x^{-1}]]$. So by the above argument, we have $\psi_2 :$ $Q[[x^{-1}]] \to Q[[x^{-1}]]$ such that $\psi_2 \neq \psi_1$ and ${}_{Q[x^{-1}]}|\psi_2|_{Q[x^{-1}]} = \phi$.

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