R[x]-LINEAR MAPS OF THE MACAULAY-NORTHCOTT MODULE

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1. Introduction

Northcott in [3] considered the module K[x⁻¹] of inverse polynomials over the polynomial ring K[x] (with K a field). The idea for this module came from Macaulay's work in [1]. McKerrow in [2] generalized Northcott's work and considered the module M[x⁻¹] over R[x] (with R a ring and M a left R-module). If E is an injective left R-module and R is left noetherian then E[x⁻¹] is an injective left R[x]-module (see [2]). In [4] and [5] we studied the behaviors of these so-called Macaulay-Northcott modules when we apply the torsion and extension functors to them. In this paper we will consider the R[x]-linear maps of these modules.

DEFINITION 1.1. Let R be a ring and M be a left R-module then M[x⁻¹] is a left R[x]-module defined by

\[ x(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = m_1 + m_2 x^{-1} + \cdots + m_n x^{-n+1}. \]

We call M[x⁻¹] a Macaulay-Northcott Module.

DEFINITION 1.2. Let C be the category of left R-module and D be the category of left R[x]-module. Let \( f : R M \to R N \) be a linear map, then \( T : C \to D \) defined by \( T(M) = M[x^{-1}] \) and \( T(f) = f \) (where \( f(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = f(m_0) + f(m_1) x^{-1} + \cdots + f(m_n) x^{-n} \)) is a functor between C and D. We call T the Macaulay-Northcott Functor.

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THEOREM 1.3. There is a natural isomorphism

\[ \text{Hom}_{R[x]}(M[x^{-1}], N[x^{-1}]) \cong \text{Hom}_{R}(M, N)[[x]]. \]

Proof. See Theorem 4.1 [4].

Suppose \( \phi : M \to N \) is a \( R \)-linear map, then we have the obvious map \( M[x^{-1}] \to N[x^{-1}] \), namely \( \phi + 0 \cdot x + 0 \cdot x^2 + \cdots \in \text{Hom}_{R}(M, N)[[x]]. \)

2. The Macaulay-Northcott Module

PROPOSITION 2.1. Let \( M \) be an essential extension of \( N \) as a left \( R \)-module then \( M[x^{-1}] \) is an essential extension of \( N[x^{-1}] \).

Proof. Let \( m_0 + m_1 x^{-1} + \cdots + m_i x^{-i} \in M[x^{-1}] \) w.l.o.g. let \( m_i \neq 0 \) then there is \( r_i \in R \), \( r_i \neq 0 \) such that \( m_i r_i \in N \), \( m_i r_i \neq 0 \). So \( r_i x_i (m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = r_i m_i \in N[x^{-1}] \). Hence \( M[x^{-1}] \) is an essential extension of \( N[x^{-1}] \).

REMARK 2.2. Let \( R \) be left noetherian. If \( E \) is an injective envelope of \( M \) then \( E[x^{-1}] \) is an injective envelope of \( M[x^{-1}] \).

Note that if \( RM \subset R N \), then

\[ \frac{N[x^{-1}]}{M[x^{-1}]} \cong \frac{N}{M}[x^{-1}]. \]

PROPOSITION 2.3. If \( 0 \to M \to E^0 \to E^1 \to \cdots \) is a minimal injective resolution of \( M \) as a left \( R \)-module then

\[ 0 \to M[x^{-1}] \to E^0[x^{-1}] \to E^1[x^{-1}] \to \cdots \]

is a minimal injective resolution.

Proof. Let \( 0 \to M \xrightarrow{\epsilon} E^0 \xrightarrow{d_1} E^1 \xrightarrow{d_2} \cdots \) and \( 0 \to M[x^{-1}] \xrightarrow{\bar{\epsilon}} E^0[x^{-1}] \xrightarrow{d_1} E^1[x^{-1}] \xrightarrow{d_2} \cdots \). Let \( m_0 + m_1 x^{-1} + \cdots + m_i x^{-i} \in M[x^{-1}] \).

Then \( d_0 \circ \bar{\epsilon} (m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = (d_0 + 0 \cdot x + 0 \cdot x^2 + \cdots ) \circ (e + 0 \cdot x + 0 \cdot x^2 + \cdots ) (m_0 + \cdots + m_i x^{-i}) = d_0 (e(m_0)) + d_0 (e(m_1)) x^{-1} + \cdots + d_0 (e(m_i)) x^{-i} = 0 \) So \( \text{im} (\bar{\epsilon}) \subset \text{ker} (d_0) \). Let \( e_0 + e_1 x^{-1} + \cdots +
\( e_i x^{-1} \in \ker(d_0) \). Then \( d(e_0) + d(e_1)x^{-1} + \cdots + d(e_i)x^{-1} = 0 \). So \( d(e_0) = d(e_1) = \cdots = d(e_i) = 0 \). So \( e_0, e_1, \ldots, e_i \in \text{im}(e) \). So there exist \( m_0, m_1, \ldots, m_i \) such that \( e(m_0) = e_0, e(m_1) = e_1, \ldots, e(m_i) = e_i \). Now \( e(m_0 + m_1 x^{-1} + \cdots + m_i x^{-i}) = e_0 + e_1 x^{-1} + \cdots + e_i x^{-i} \). So \( \text{im}(e) = \ker(d_0) \). By the same process we have \( \text{im}(d_k) = \ker(d_{k+1}) \). And by Remark 2.2, \( E^{k+1}[x^{-1}] \) is an injective envelope of \( E^k[x^{-1}] \). So \( 0 \to M[x^{-1}] \to E^0[x^{-1}] \to \cdots \) is a minimal injective resolution of \( M[x^{-1}] \) as a left \( R[x] \)-module.

**PROPOSITION 2.4.** Let \( \phi : M[x^{-1}] \) be \( R[x] \)-linear map.

Then \( \phi(M) \subset M \).

**Proof.** Suppose \( m \in M \) and \( \phi(m) = f \notin M \) and \( f = m_0 + m_1 x^{-1} + \cdots + m_n x^{-n} \). Then for \( x \in R[x] \), \( \phi(xm) = 0 \) and \( x\phi(m) = m_1 + m_2 x^{-1} + \cdots + m_n x^{-n} \neq 0 \) So \( x\phi(m) \neq \phi(xm) \). This contradicts the fact that \( \phi \) is a \( R[x] \)-linear map. So \( \phi(M) \subset M \).

**PROPOSITION 2.5.** Let \( \phi : M[x^{-1}] \to M[x^{-1}] \) be \( R[x] \)-linear map.

1) If \( M \xrightarrow{\phi} M \) is one to one so is \( \phi \).

2) If \( M \xrightarrow{\phi} M \) is an isomorphism so is \( \phi \).

**Proof of 1).** Suppose \( \phi \) is not one to one. Let \( h = \ker(\phi) \) for \( h = m_0 + m_1 x^{-1} + \cdots + m_n x^{-n} \) and w.l.o.g. \( m_n \neq 0 \). Then \( \phi(h) = \phi(m_0 + m_1 x^{-1} + \cdots + m_n x^{-n}) = 0 \). Since \( \phi \) is \( R[x] \)-linear map, \( x\phi(m_0) + x\phi(m_1 x^{-1}) + \cdots + x\phi(m_n x^{-n}) = \phi(m_1 + m_2 x^{-1} + \cdots + m_n x^{-n+1}) = 0 \). Multiply \( x \) on the left hand side again, then we have \( \phi(m_2) + \phi(m_3 x^{-1}) + \cdots + \phi(m_n x^{-n+2}) = 0 \). Repeat this process until we have \( \phi(m_n) = 0 \). So \( m_n \in \ker(M|\phi|M) \). This contradicts the fact that \( M|\phi|M \) is 1-1. So \( \phi \) is 1-1.

**Proof of 2).** Suppose \( M|\phi|M \) is an isomorphism. Then by 1) \( \phi \) is one to one so we want to show \( \phi \) is onto. Let \( f \in M[x^{-1}] \) and \( f = m_0 + m_1 x^{-1} + \cdots + m_i x^{-i} \). Suppose \( \phi(g) = f \) for \( g \in M[x^{-1}] \). Then \( x^i \phi(g) = m_i \). So let \( g = n_0 + n_1 x^{-1} + \cdots + n_i x^{-i} \). Then \( x^i g = n_i \). Now choose \( n_i \) such that \( \phi(n_i) = m_i \). Let \( \phi(n_i x^{-1}) = c_{i-1} + m_i x^{-1} \). Choose \( n_{i-1} \) such that \( \phi(n_{i-1}) = m_{i-1} - c_{i-1} \). And let \( \phi(n_{i-1} x^{-1}) + \phi(n_i x^{-2}) = c_{i-2} + m_{i-1} x^{-1} + m_i x^{-2} \). Choose \( n_{i-2} \) such that \( \phi(n_{i-2}) = n_{i-2} - c_{i-2} \). By this process we can get \( n_{i-3}, \ldots, n_0 \) and we have \( \phi(g) = f \). So \( \phi \) is onto. So \( \phi \) is an assumption.
**Proposition 2.6.** \( \sigma : M[[x^{-1}]]/M[x^{-1}] \to M[[x^{-1}]]/M[x^{-1}] \) by 
\[ f + M[x^{-1}] \to x(f + M[x^{-1}]) \] is an isomorphism.

**Proof.** Let \( f + M[x^{-1}] \in \ker(\sigma) \) and let \( f = a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots \). Then \( \sigma(f + M[x^{-1}]) = x(f + M[x^{-1}]) = M[x^{-1}] \). So \( f + M[x^{-1}] = M[x^{-1}] \). So \( f \) is one to one. Let \( f + M[x^{-1}] = (a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots) + M[x^{-1}] \in M[[x^{-1}]]/M[x^{-1}] \). Let \( g + M[x^{-1}] = (a_0 x^{-1} + a_2 x^{-2} + a_2 x^{-3} + \cdots) + M[x^{-1}] \), then \( \sigma(g + M[x^{-1}]) = f + M[x^{-1}] \). Hence \( \sigma \) is onto. So \( \sigma \) is an isomorphism.

**Theorem 2.7.** Let \( \phi : E[x^{-1}] \to E[x^{-1}] \) be a linear map for \( RE \) injective, then there is a \( \psi : E[[x^{-1}]] \to E[[x^{-1}]] \) such that \( E[[x^{-1}]][\psi]|_{E[x^{-1}]} = \phi \). Moreover \( \psi \) is not unique in general.

**Proof.** Since \( E[[x^{-1}]] \) is an injective left \( R[x] \)-module we can complete the following diagram

\[
\begin{array}{ccc}
E[x^{-1}] & \hookrightarrow & E[[x^{-1}]] \\
\phi \downarrow & \nearrow \psi \\
E[[x^{-1}]] & &
\end{array}
\]

So we have \( \psi \) such that \( E[x^{-1}][\psi]|_{E[x^{-1}]} = \phi \).

Let \( \psi_1, \psi_2 : E[[x^{-1}]] \to E[[x^{-1}]] \) and \( E[x^{-1}][\psi_i]|_{E[x^{-1}]} = \phi \) for \( i = 1, 2 \). Then \( \psi_1|x_{E[x^{-1}]} = \phi \), \( \psi_2|x_{E[x^{-1}]} = \phi \) and \( \psi_1 - \psi_2|_{E[x^{-1}]} = 0 \). So \( E[x^{-1}] \subset \ker(\psi_1 - \psi_2) \). So we have an induced map

\[
E[[x^{-1}]])/E[x^{-1}] \to E[[x^{-1}]] \text{ by } f + E[x^{-1}] = (\psi_1 - \psi_2)(f).
\]

Now consider the following. Let \( \phi : E[x^{-1}] \to E[x^{-1}] \) be a linear map. Let \( \psi_1 : E[[x^{-1}]] \to E[[x^{-1}]] \) such that \( E[x^{-1}][\psi_1]|_{E[x^{-1}]} = \phi \). Let \( \sigma : E[[x^{-1}]]/E[x^{-1}] \to E[[x^{-1}]] \) be a non zero linear map, then there is a non zero linear map \( \tau : E[[x^{-1}]] \to E[[x^{-1}]] \) such that \( \tau(f) = \sigma(f + E[[x^{-1}]]) \) and \( E[x^{-1}] \subset \ker(\tau) \). Let \( \psi_2 : E[[x^{-1}]] \to E[[x^{-1}]] \) such that \( \psi_2 = \psi_1 - \tau \), then

\[
E[x^{-1}][\psi_2]|_{E[x^{-1}]} = E[x^{-1}][\psi_1 - \tau]|_{E[x^{-1}]} = E[x^{-1}][\psi_1]|_{E[x^{-1}]} = \phi.
\]

So there is \( \psi_2 \) such that \( \psi_1 \neq \psi_2 \).
Example 2.8. Let \( R = \mathbb{Z} \) and \( E = \mathbb{Q} \). Let \( \psi_1 : \mathbb{Q}[x^{-1}] \to \mathbb{Q}[x^{-1}] \) be a linear map. Let \( \phi : \mathbb{Q}[x^{-1}] \to \mathbb{Q}[x^{-1}] \) be a linear map such that \( \mathbb{Q}[x^{-1}] \phi \mathbb{Q}[x^{-1}] = \phi \). Consider \( \mathbb{Q}[x^{-1}]/\mathbb{Q}[x^{-1}] \) and \( \mathbb{Q}[x^{-1}] \) as left \( \mathbb{Z}[x] \)-modules. Let \( f = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots \in \mathbb{Q}[x] \). For any non-zero \( g \in \mathbb{Z}[x] \) we claim that \( g \cdot f \notin \mathbb{Q}[x] \). Suppose \( g \cdot f = h \in \mathbb{Q}[x] \) and w.l.o.g. \( \text{deg} h = n \). Then \( h^{(n+1)}(x) = 0 \), but \( (g \cdot f)^{(n+1)}(x) \neq 0 \). Let \( f = 1 + x^{-1} + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \in \mathbb{Q}[x^{-1}] \). For non-zero \( g \in \mathbb{Z}[x] \) we claim \( g \cdot f \notin \mathbb{Q}[x^{-1}] \). Let \( g = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \). Then \( (a_1 + a_{n-1} x^{-1} + \cdots + a_0 x) \cdot f \) and \( (a_1 + a_{n-1} x^{-1} + \cdots + a_0 x) \cdot f \) have some coefficient for each \( x^{-n} \) and \( x^n \) terms. So \( (a_1 + a_{n-1} x^{-1} + \cdots + a_0 x) \cdot f \notin \mathbb{Q}[x^{-1}] \). So \( \psi_2 : \mathbb{Q}[x^{-1}] \to \mathbb{Q}[x^{-1}] \) such that \( \psi_2 \neq \psi_1 \) and \( \mathbb{Q}[x^{-1}] \psi_2 \mathbb{Q}[x^{-1}] = \phi \).

References

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