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ON METRIZABILITY OF MOORE SPACES

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0.Introduction

Since the starting of the 20^{th} century the various topologists has been devoted in a study in finding topological charterizations of metric spaces(see [4],[13],and [14]).

This note is study of some condition which a topological space is a Moore space and the space would be metrizable under a somewhat considerable arguments. Moreover, it seems to us to be worthwhile to discuss in some detail this little-modificated metrizable theorem and some of its postulations.

In §1 we introduce the notion of Moore space and show that every metric space is a Moore space. In §2 we prove that a regular T_2-space which is countably compact is a Moore space. §3 and §4 the little-modificated metrizable theorem for the Moore apaces are introduced. Finally, in §5 we prove that the regular T_2-space with a uniform base is a Moore space and hence is metrizable and that every regular space having the properties of the locally compact and locally connected with a G_δ-diagonal has a θ-separating cover and hence is metrizable.

1.F.Burton Jones's Theorem

Theorem 1. Let X be regular topological space(Regular $[T_2]$). Suppose that there exists a countable family Δ of open covers of X such that G and H are disjoint closed subsets of X, one of which is compact, there is an element $\mathcal{G}$ of Δ such that no element of $\mathcal{G}$ intersects both G and H(i.e.,$\mathcal{G}^*(G) \cap H = \emptyset = \mathcal{G}^*(H) \cap G$). Then X is metrizable [6].

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While this is not precisely the form in which Moore stated the theorem [13], we are stating it in this form to emphisize the kind of normality (or Metrizability of Moore space with separating open covers in §6) contained in the hypothesis that the sets, $\mathcal{G}^*(p)$ and $\mathcal{G}^*(H)$ are disjoint open subsets of X containing P and H(closed in X) respectively. It is also easy to see why one would expect the theorem to be true.

For suppose that X is metric and that let $S_n = \{S(x,\delta)|x \in X, \delta = \frac{1}{n}, n = 1,2,\cdots\}$. Clearly S_n covers X and if $\frac{1}{n} < d(P,H)/2$ then $\mathcal{G}^*(P) \cap \mathcal{G}^*(H) = \emptyset$ by the triangle inequality for the metric distance function d. Futhermore for disjoint closed sets G and H that $\mathcal{G}^*(G) \cap \mathcal{G}^*(H) = \emptyset$ because at to know this is not true for some of the simplest metric apaces(e.g. the plane R^2) no matter how $\mathcal{G}$ is defined. Of course, if $d(H,G)$ were positive, the above argument would work and this would be the case if either G or K were compact.

But for this situation a slightly stronger form of Moore theorem is possible.

The other from of Moore's Metrization theorem.

Theorem 2. Let X be a regular topological space(regular $[T^2]$). Suppose that there exists $\Delta = \{\mathcal{G}_n|n \in N, \mathcal{G}_n : \mathit{open\ cover\ of} X\}$ such that if G and H are disjoint clased subsets of X, one of which is countably compact, there is an element $\mathcal{G}$ of Δ such that no element of $\mathcal{G}$ intersects both G and H(i.e.,$\mathcal{G}^*(G) \cap H = \emptyset = \mathcal{G}^*(H) \cap G$). Then X is metrizable.

Proof. Let $\mathcal{A}_1, \mathcal{A}_2, \cdots$ be a simple well-ordering of Δ^* and let $\mathcal{G}_n$ refine $\mathcal{A}_i$ for $i < n$ such that for each $n \in N$, each element of $\mathcal{G}_n$ is a subset of some elements of $\mathcal{A}_i$ for $1 \leq i \leq n$. Suppose that X is not metrizable ; then the countable family $\Delta = \{\mathcal{G}_n|n \in N\}$ must fail to satisfy the hypothesis of §1. When $\{\mathcal{G}_n\}$ is substituted for Δ^*. Hence there exist a closed set G and a point P of G^c such that for each n, $\mathcal{G}^*(P) \cap \mathcal{G}^*(G) \neq \emptyset$.

For each n, let P_n denote a point in this intersection and let H denote the closure of the set of all these points. Since $\{P\}$ is closed and countably compact, the sequence $< P_n >$ converges to P. Hence G and $H \setminus G$ are disjoint closed sets and $H \setminus G$ is countably compact; so there must exist a natural number n such that no element of $\mathcal{A}_n$ intersets both. Hence the same must be true of each of $\mathcal{G}_n . \mathcal{G}_{n+1}, \cdots$

which is a contradiction since for i large enough for P_i to belong to $H \setminus G$, $\mathcal{G}_i^*(G)$ obviously intersects both G and $H \setminus G$.

The Theorem follows from this controduction.

3.Moore Spaces

In this section we use the techniques developed by Jones [6] and [13] to abtain a generalization of the above theorem 2. In the light of the usefulness of the concept of a "countably compact" in the study of a Moore space, we are about to prove the main theorem in §5.

Above of all we introduce the fact the topological generalization of metric spaces discovered by Moore [14] as follows;

Definition 1. Let X be a regular topological space. Suppose that there exist $\triangle = \{\mathcal{G}_n | n \in N\}$, the collection of open covers of X such that if G is a closed subset of X and P is a point G^c, then there is a cover $\mathcal{G}_{n_0}$ in $\triangle$ such that no element of $\mathcal{G}_{n_0}$ contains P and intersects $G(say, G \cap st(P, \mathcal{G}_{n_0}) = \emptyset)$. Such a space is called a Moore space and $\triangle$ is called a development for X. Clearly all metric space are Moore space, for as before in §1, if for $\mathcal{G}_n = \{S(p,\delta) | p \in X, \delta = \frac{1}{n}, n \in N\}$ in a metric space X, $\mathcal{G}_n$ covers the space X and the family $\{\mathcal{G}_n\}$ has all of the properties require of $\triangle$ in §1. And it is easy speculate that perhaps all Moore spaces are metric spaces.

It is well known that in a great many situations what is true of closed and finite point sets is true of closed and countably compact point sets and conversely. So substituting "compact" for "countably compact" as following(again let X denote a regular $[T_2]$);

Theorem 3. Suppose that there exists a countable family $\triangle$ of open covers of X such that if G and H are disjoint closed subsets of X, one of which is countably compact, then there is an element $\mathcal{G}$ of $\triangle$ such that no element of $\mathcal{G}$ intersections both G and H $(say, \mathcal{G}^*(G) \cap H = \emptyset = \mathcal{G}^*(H) \cap G)$. Is every space of this sort metric ?

Proof. It is readily verified that every Moore space is such a space for it is no loss of generality to suppose that the development $\triangle$ postulated to exist for a Moore space has a simple well-ordering,$\mathcal{G}_1, \mathcal{G}_2, \cdots$ and that $\mathcal{G}_n$, refines $\mathcal{G}_i$ for $i < n$; Hence,X being a Moore space, for each point P of X there exists a covers $\mathcal{G}$ in $\triangle$ such that $\mathcal{G}^*(P) \cap G = \emptyset$.

When X is countably compact, this set of covers is finite and which ever one has the largest subscript(in the well-ordering) has the required property, namely, no element of it intersections both G and H. Thus, one may see that every Moore space is a space of the above sort. But not every Moore space is metric ; so not every space of this sort is metric despite the strength of the "countably compact=finite(refinement)" principle.

[Example of Nonmetrizable Moore spaces][13] The best-known example of a Moore space which is not metrizable is due to R.L.Moore. Let $X = \{(x,y)|y \geq 0 \ in \ R^2\}$ and let base elements be of two types (1) if a circle lies entirely above the X-axis its interior is a region and (2) if a circle is tangent to the X-(from above) then its interior plus the point of tangency is a region. Let $\mathcal{G}_n(n \in N)$ be the collection of all regions of diameter $\frac{1}{n}$(d:the usual metric).

The family $\Delta = \{\mathcal{G}_n|n \in N\}$ the properties decomanded of Δ to make X a Moore space. Since no region contains two points of X belonging to the X^-,the X^- has become a descrete point set. If X were meterizable, X would necessarily have a countable basis(the part of X above the X-axis is still separable just as it was in the plane) and every countable basis would leave uncountably many points on the X-axis uncovered. Hence X is not metrizable.

4. Metrization Of Moore Spaces

Since a Moore space is not necessarily metrizable, the question arises as to just how a Moore space differs from a metric space. Many years ago Jones [4] thought that the property of normality was the only difference(and this is indeed the case if same countable set is dense in the space [7]). But despite the efforts of many investigatiors this problem is still unresolved. At present the best answer to the question is paracompactness that every paracompact Moore space is metric was first observed by Bing [12]. We give below a somewhat different argument.

Theorem 4. Let X be a regular metacompact Moore space, then X is metrizable.

Proof. Suppose that $\Delta = \{\mathcal{G}_n|n \in N\}$ is a development for X. Since X is regular let us assume that Δ has property that not only does $\mathcal{G}_{n+1}$ refine $\mathcal{G}_n$ but that the collection of closures of elements of $\mathcal{G}_{n+1}$ does also (i.e , if G belong to $\mathcal{G}_{n+1}$ then $\bar{G}$ is a subset of at least one element of $\mathcal{G}_n$). Since X is metacompact, let us also assume that for each natural number n,$\mathcal{G}_n$ is locally finite. Let G and H denote disjoint closed subsets of X, such that one of tham, say H, is countably compact and for each n, let $\mathcal{G}_n$(H) denote the collection of all element of $\mathcal{G}_n$ which intersect H. Since H is countably compact and $\mathcal{G}_n$ is locally finite($\mathcal{G}_n$, countably open covering of X, contains a finite subcollection, covering X), $\mathcal{G}_n$ contains a finite subcollection covering X,$\mathcal{G}_n$ is finite. Now if for each n, $\mathcal{G}_n^*(G) \cap H \neq \emptyset$, there exists a sequence $H_1.H_2.H_3.\cdots$ such that $H_1 \supset \overline{H_2} \supset H_2 \supset \cdots$ and for each n, H_n is an element of $\mathcal{G}_n(H)$ which intersects both G and H. Let p be a point of $\cap(H, \bar{H}_n)$, then P is a point of H which for each n belongs to H_n. So for each n, $\mathcal{G}_n^*(P) \cap G \neq \emptyset$ contrary to the required behavior of development for the Moore space X Consequently $\triangle = \{\mathcal{G}|n \in N\}$ has the properties of $\triangle$ required in §2 to make X metrizable.

5. Main Theorems

Lemma 1.[8] A space X has a G-diagonal if and only if there exists a sequence of open covers $\{\mathcal{G}|n \in N\}$ of X such that, for each $x, y \in X (x \neq y)$ implies there exists m such that $y \notin st(x, \mathcal{G}_m)$.

With the aid of the above Lemma, we can show the theorem as follows;

Theorem 5. Let X be a regular metacompact w$\triangle$-space with a G_δ-diagonal. Then X is regular T_2-space with uniform base and hence X is point-wise metacompact Moore space.

Proof. Since X is a w$\triangle$-space and has a G_δ-diagonal, there exists a sequence $\triangle = \{\mathcal{G}_1, \mathcal{G}_2, \cdots\}$ of covers of X such that if $x \neq y$, then $y \notin st(x, \mathcal{G}_n)$ for some n and for each $x_0 \in X$, if $x_n \in st(x, \mathcal{G}_n)$ for $n = 1, 2, \cdots$ then the sequence $< x_n >$ has a cluster point. For each n, let $\mathcal{G}_{n+1}$ be a point-finite open refinement of $\mathcal{G}_n$ such that $\{\overline{B}|B \in \mathcal{G}_{n+1}\}$ is a refinement of $\mathcal{G}_n$(X:regular).

Letting $\mathcal{G} = \cup_1^\infty \mathcal{G}_n$, we show that $\mathcal{G}$ is a uniform base for X, and now let $\mathcal{G}(x) = \{st(x, \mathcal{G})|x \in X, n \in N\}$ then it suffices to show that $\mathcal{G}(X)$ is a neighborhood base at x for each $x \in X$. For all $U \subset X$, open subset containg X, assume that for all n, $st(x, \mathcal{G}_n) \not\subset U$. Then there exists $x_n \in st(x, \mathcal{G}_n) \setminus U$ for each n, and thus the sequence $\{x_n\}$ has a cluster point $y \in X (y \neq x)$. Since $st(x, \overline{\mathcal{G}_{n+1}}) \subset st(x, \mathcal{G}_n)$ for each n, we get that $y \in st(x, \overline{\mathcal{G}_{n+1}}) \subset st(x, \mathcal{G}_n)$ for every n. It's not true that y belong to $st(x, \mathcal{G}_n)$. And then in [5], Jones show that a regular Hausdorff space X has a uniform base it is necessary and sufficient that X be a point-wise paracompact Moore space.

Finally, in [12] R.H.Bing proved that the metacompact Moore space is metrizable.

In the above sections we have proved that in the regular T_2-space, every contably compact Moore space is metrizable. Then, what is relevant condition for the Moore space ?

First we state some revelent and basic Lemma and theorems ;

Lemma 2.[14] Let X be a space with a θ-separating cover. If every closed subset of X is a G_δ-set then X has G_δ-diagonal .

Lemma 3.[3] Every regular wΔ-space with a G^*_δ-diagonal is a Moore space.

Theorem 6. Every regular space having the properties of the locally compact and locally connected with a G_δ-diagonal has a θ-separating cover.

Proof. Let υ be any open cover of X and $\Delta = \{\mathcal{G}_n | n \in N\}$ be a collection of open covers of X such that for each n,$\mathcal{G}_n$ is a refinement of υ (i.e.,each element of $\mathcal{G}_n$ is a subset of some element of υ). Since X is locally compact, for all $x \in X$, there is a neighbourhood compact V(x) such that $V(x) = st(x, \mathcal{G}_{n_0})$ for some n_0, and since X is locally connected, $st(x, \mathcal{G}_{n_0})$ contains a connected open set containing $x \in X$. Consequently Δ is an open refinements of υ with the requirements of θ-refinable for X and it exhibits the G_δ-diagonal property for X. For each $n \in N$, let $\mathcal{A}_{n_1}, \mathcal{A}_{n_2}, \cdot\ \cdot$ be a θ-refinement of $\mathcal{G}_n$ then $\{\mathcal{A}_{n_k} | n \in Nk \in N\}$ is a θ-separating cover of X.

For the sake of the construction of Moore space, the following Lemma, due to Burk, should be introduced ;

Lemma 4.[14] Let X be a regular, θ-refinable wΔ-space. Then there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open covers of X such that for each $x \in X$,

(a) $C_x = \cap_1^\infty st(x, \mathcal{G}_n)$ is compact ;

(b) $\{st(x, \mathcal{G}_n) | n \in N\}$ is a base for C_x.

Proof. Let $v_1, v_2, v_3, \cdots$ be a wΔ-sequence for X. By introduction on n construct for each positive integer n a sequence $\mathcal{A}_{n_1}, \mathcal{A}_{n_2}, \cdots$ of open covers of X such that (1) for $k = 1, 2, \cdots, \{\bar{A} | A \in \mathcal{A}_{n_k}\}$ refines v_n and A_{ij}, $1 \leq i \leq n-1, 1 \leq j \leq n-1$; (2)for each $x \in X$ there is a $k \in N$ such that ord$(x, \mathcal{A}_{n_k})$ is finite. For $n \in N$, let $\mathcal{G}_n = \mathcal{A}_{n_1}$. Then $\mathcal{G}_1, \mathcal{G}_2, \cdots$ satisfies properties (a) abd (b).

By R.E.Hodel [14], X has a θ-separating cover and from main Theorem 6, every closed subset of X is a G_δ-set. It follows by Lemma 2 that has G_δ-diagonal and hence by Lemma 3, X is a Moore space hence the following theorem is clear ;

Theorem 7. *Let X be a regular, locally compact, locally connected and wΔ-space with a point-countable separating open cover. Then X is a Moore space.*

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