AN IDEAL CHARACTERIZATION OF
COMMUTATIVE BCI-ALGEBRAS

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In our joint paper [6], the author of this paper and X. L. Xin introduced the concept of commutative BCI-algebras as a generalization of one of commutative BCK-algebras and a number of important properties of it were obtained. C. S. Hoo [2] also introduced the same concept called pseudo-commutative BCI-algebras. In my note [5], I introduced the concept of commutative ideals in BCK-algebras, by means of which some characterizations of commutative BCK-algebras were obtained. Now the results in [5] will be generalized to BCI-algebras. We define commutative ideals and show that a BCI-algebra is commutative if and only if every closed ideal of it is commutative. Moreover, we obtain an interesting results that let I be a commutative ideal and let A be a closed ideal, if I ⊆ A then A is also a commutative ideal.

Let us recall some definitions and results which are necessary for development of this paper.

By a BCI-algebra is meant a set X with a binary operation * and a constant 0 on it satisfying the following axioms:

(I) \((x * y) * (x * z) \leq z * y,\)
(II) \(x * (x * y) \leq y,\)
(III) \(x \leq x,\)
(IV) \(x \leq y \text{ and } y \leq x \text{ imply } x = y,\)
(V) \(x \leq y \text{ if and only if } x * y = 0.\)

A BCI-algebra satisfying
(VI) \(0 \leq x\)

is called a BCK-algebra.

For any BCI-algebra X, the following hold:

(1) \(x \leq 0\) implies \(x = 0,\)
(2) \(x \cdot 0 = x,\)
(3) \((x * y) * z = (x * z) * y,\)
(4) \(x * (x * (x * y)) = x * y,\)
(5) \(0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y),\)
(6) \((x + z) \ast (y + z) \leq x + y,\)
(7) \(x \leq y \text{ implies } x + z \leq y + z \text{ and } z \ast y \leq z \ast x.\)

A nonempty subset \(I\) of \(X\) is said to be an ideal if it satisfies:

(i) \(0 \in I,\)
(ii) \(x \ast y \in I\) and \(y \in I\) imply \(x \in I.\)

We gave an equivalent condition of ideals which is needed for this paper.

**Proposition 1 ([4]).** A nonempty subset \(I\) of \(X\) is an ideal if and only if it satisfies (i) and

(iii) \(y, z \in I\) and \(x \ast y \leq z\) imply \(x \in I\) for all \(x, y, z\) in \(X.\)

Any ideal \(I\) have the following property:

(iv) \(x \in I\) and \(y \leq x\) imply \(y \in I.\)

For more details of BCI-algebras and its ideals we refer the readers to the references [1] and [2] listed in the end of this paper.

In this paper, unless otherwise specified, \(X\) will always mean a BCI-algebra.

**Definition 1.** A nonempty subset \(I\) of \(X\) is called a commutative ideal if it satisfies (i) and

(v) \((x \ast y) \ast z \in I\) and \(z \in I\) imply \(x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in I\)
for all \(x, y, z\) in \(X.\)

Obviously, every ideal of a p-semisimple BCI-algebra is commutative([3]).

**Theorem 2.** Any commutative ideal must be an ideal.

*Proof.* Let \(I\) be a commutative ideal of \(X\) and let \(x \ast z \in I\) and \(z \in I,\) then \((x \ast 0) \ast z \in I\) and \(z \in I.\) From (v) it follows that \(x = x \ast ((0 \ast (0 \ast x)) \ast (0 \ast (0 \ast (x \ast 0)))) \in I,\) so \(I\) an ideal.

**Remark.** An ideal may not be commutative. For instance, \(X = \{0, 1, 2, 3, 4\},\) the operation \(\ast\) is given by the following table:
An ideal characterization of commutative BCI-algebras

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Then $(X; *, 0)$ is a BCI-algebra, $\{0, 1\}$ is an ideal of $X$ but not commutative.

**Theorem 3.** An ideal $I$ is commutative if and only if it satisfies

(vi) $x * y \in I$ implies $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I$.

**Proof.** This is routine and omitted.

**Definition 2** ([1]). An ideal $I$ is closed if $0 * z \leq \emptyset$ whenever $x \in I$ for all $x$ in $X$.

For a closed ideal the condition (vi) have a simpler form.

**Theorem 4.** Let $I$ be a closed ideal of $X$, then $I$ is commutative if and only if it satisfies

(vii) $x * y \in I$ implies $x * (y * (y * x)) \in I$.

**Proof.** Suppose $I$ is commutative and let $x * y \in I$, then $0 * (x * y) \in I$ as $I$ is closed and by (vi) we have $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I$.

Since

$$(x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y)))))$$

$$(y * (y * x)) * (0 * (0 * (x * y)))) * (y * (y * x))$$

$= 0 * (0 * (0 * (x * y)))$$

$= 0 * (x * y) \in I,$

it follows from Proposition 1 that $x * (y * (y * x)) \in I$, that is, $I$ satisfies (vii).

Conversely, if $I$ satisfies the condition (vii) and $x * y \in I$ then $x * (y * (y * x)) \in I$. Note that $0 * (0 * (x * y)) \in I$ by (vi). Since

$$(x * ((y * (y * x)) * (0 * (0 * (x * y))) * (x * (y * (y * x))))$$

$$(y * (y * x)) * ((y * (y * x)) * (0 * (0 * (x * y))))$$

$0 * (0 * (x * y)) \in I,$
by Proposition 1 we have $x \ast ((y \ast (y \ast x)) \ast (0 \ast (0 \ast (x \ast y)))) \in I$, so $I$ is commutative. The proof is complete.

With respect to the distribution of commutative ideals we have

**Theorem 5.** Given two ideals $I$ and $A$ of $X$ with $I \subset A$, if $A$ is closed and if $I$ is commutative then $A$ is also commutative.

**Proof.** Assume that $x \ast y \in A$. For simplicity of notation let us write $u = x \ast y$, thus $0 \ast u \in A$ as $A$ is closed. Since $I$ is commutative and $(x \ast u) \ast y = 0 \in I$, from (vi) it follows

$$(x \ast u) \ast (y \ast (x \ast u)) = (x \ast y) \ast ((y \ast (y \ast (x \ast u))) \ast (0 \ast (0 \ast ((x \ast u) \ast y)))) \in I.$$ 

Note $I \subset A$, we have

$$(x \ast u) \ast (y \ast (y \ast (x \ast u))) \in A,$$

i.e.,

$$(x \ast (y \ast (y \ast (x \ast u)))) \ast u \in A.$$ 

Combining $u \in A$ we obtain $x \ast (y \ast (y \ast (x \ast u))) \in A$. As

$$
\begin{align*}
(x \ast (y \ast (y \ast x))) \ast (x \ast (y \ast (y \ast (x \ast u)))) & \\
\le (y \ast (y \ast (x \ast u))) \ast (y \ast (y \ast x)) & \\
\le (y \ast x) \ast (y \ast (x \ast u)) & \\
\le (x \ast u) \ast x & \\
= 0 \ast u \in A,
\end{align*}
$$

using Proposition 1 we have $x \ast (y \ast (y \ast x)) \in A$. This says that $A$ is commutative. This finishes the proof.

**Definition 3** ([6]). A BCI-algebra $X$ is called commutative if, for all $x, y$ in $X$,

$$
(8) \ x \leq y \text{ implies } x = y \ast (y \ast x).
$$

**Proposition 6** ([6]). A BCI-algebra $X$ is commutative if and only if it satisfies

$$
(9) \ x \ast (x \ast y) = y \ast (y \ast (x \ast y)).
$$

Next we are ready to prove an important result.
Theorem 7. For any BCI-algebra $X$, the following are equivalent:

(10) $X$ is commutative,
(11) every closed ideal of $X$ is commutative,
(12) the zero ideal $\{0\}$ is commutative.

Proof. (10) $\Rightarrow$ (11) Suppose $X$ is commutative and $I$ is a closed ideal of $X$. If $x \cdot y \in I$ then $0 \cdot (x \cdot y) \in I$. By means of Proposition 6 we have

\[(x \cdot (y \cdot (y \cdot x))) \cdot (x \cdot y) = (x \cdot (x \cdot y)) \cdot (y \cdot (y \cdot x))
= (y \cdot (y \cdot (x \cdot (y \cdot x)))) \cdot (y \cdot (y \cdot x))
= (y \cdot (y \cdot (y \cdot x))) \cdot (y \cdot (x \cdot (y \cdot x)))
= (y \cdot x) \cdot (x \cdot (x \cdot y))
\leq (x \cdot (x \cdot y)) \cdot x
= 0 \cdot (x \cdot y) \in I.
\]

Using Proposition 1 we get $x \cdot (y \cdot (y \cdot y)) \in I$. This says that $I$ is commutative. (11) holds.

(11) $\Rightarrow$ (12) It is immediate since $\{0\}$ is a closed ideal.

(12) $\Rightarrow$ (10) If $x \leq y$ then $x \cdot y = 0 \in \{0\}$. In virtue of Theorem 4 we know that $x \cdot (y \cdot (y \cdot x)) \in \{0\}$, that is, $x \cdot (y \cdot (y \cdot x)) = 0$. On the other hand, by (II) we have $(y \cdot (y \cdot x)) \cdot x = 0$, hence $x = y \cdot (y \cdot x)$. This shows that $X$ is commutative. (10) holds. The proof is complete.

Finally we discuss a quotient algebra of a BCI-algebra via a commutative ideal.

Suppose $X$ is a BCI-algebra and $I$ an ideal of $X$. An equivalent relation $\sim$ on $X$ is defined by putting $x \sim y$ if and only if $x \cdot y \in I$ and $y \cdot x \in I$, we denote the equivalent class containing $x$ by $C_x$ and let $X/I = \{C_x : x \in X\}$, then $(X/I; \cdot, C_0)$ is a BCI-algebra, where

$C_x \cdot C_y = C_{x \cdot y}$ for all $x, y$ in $X$. If $I$ is also closed then $C_0 = I$; otherwise $C_0 \neq I$.

Theorem 8. Let $I$ be a closed ideal of $X$, then $I$ is commutative if and only if $(X/I; \cdot, C_0)$ is a commutative BCI-algebra.

Proof. Suppose $I$ is a closed commutative ideal of $X$. If $C_x \cdot C_y = C_0$ then $x \cdot y \in I$. By Theorem 4 we obtain $x \cdot (y \cdot (y \cdot x)) \in I$, hence
\[ C_x \star (C_y \star (C_y \star x)) = C_x \star (y \star (y \star x)) = I = C_0 \in \{C_0\}. \]
This says that the zero ideal \(\{C_0\}\) is commutative for \(X/I\). Therefore by Theorem 7, \((X/I; \star, C_0)\) is commutative.

Conversely, let \(X/I\) be commutative, then by means of Theorem 7, \(\{C_0\}\) is a commutative ideal of \(X/I\). If \(x \star y \in I\) then \(C_x \star C_y = C_{x \star y} = I = C_0 \in \{C_0\}\), and so \(C_{x \star (y \star (y \star x))} = C_x \star (C_y \star (C_y \star C_z)) \in \{C_0\}\). This says that \(x \star (y \star (y \star x)) \in C_0 = I\). From Theorem 4 it suffices to show that \(I\) is a commutative ideal of \(X\). This completes the proof.

**Lemma 9.** Suppose \((X; \star, 0)\) and \((X'; \star', 0')\) are two BCI-algebras and let \(f : X \rightarrow X'\) be a homomorphism, then \(\text{Ker}(f)\) is a closed ideal of \(X\).

**Proof.** We have know that \(\text{Ker}(f)\) is an ideal of \(X\). In order to prove that it is closed, assume \(x \in \text{Ker}(f)\) then \(f(x) = 0'\). Since \(f(0 \star x) = f(0) \star f(x) = 0' \star 0' = 0'\), it follows that \(0 \star x \in \text{Ker}(f)\), namely \(\text{Ker}(f)\) is a closed ideal of \(X\), proving this lemma.

**Corollary 10.** Let \(f : X \rightarrow X'\) be an epimorphism, then \(\text{Ker}(f)\) is a commutative ideal of \(X\) if and only if \(X'\) is a commutative BCI-algebra.

**Proof.** The quotient algebra \(X/\text{Ker}(f)\) is isomorphic to \(X'\) and use Theorem 8 and Lemma 9.

**References**


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