FUZZY PRIME IDEALS IN Γ-RINGS

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In [14], Zadeh introduced the notion of a fuzzy subset μ of a set S as a function from S into [0, 1]. Rosenfeld [12] applied this concept to the theory of groupoids and groups. Kuroki [7, 8] has studied fuzzy ideals, fuzzy bi-ideals and fuzzy semiprime ideals in semigroups. Liu [9], Mukherjee and Sen [10] and Swamy and Swamy [13] have studied fuzzy ideals and fuzzy prime ideals of a ring.

This paper is a continuation of [5] and [6]. It was shown in [6] that a Γ-homomorphic image of a fuzzy ideal which has the sup property is a fuzzy ideal. But it holds without assuming the sup property. We first prove that a Γ-homomorphic image of a fuzzy ideal is also a fuzzy ideal without assuming the sup property. In [5], the first author investigated the fuzzy prime ideals in Γ-rings. Secondly, we study more properties on fuzzy prime ideals of Γ-rings.

DEFINITION 1 ([1]). If $M = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ are additive abelian groups, and for all $x, y, z$ in $M$ and all $\alpha, \beta$ in $\Gamma$, the following conditions are satisfied

1. $x\alpha y$ is an element of $M$,
2. $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
3. $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then $M$ is called a Γ-ring.

Through this paper $M$ and $M'$ denote Γ-rings, and $0_M$ and $0_{M'}$ denote the zero elements of $M$ and $M'$ respectively.

DEFINITION 2 ([1]). A subset $A$ of $M$ is a left (right) ideal of $M$ if $A$ is an additive subgroup of $M$ and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\} (A\Gamma M)$$

is contained in $A$. If $A$ is both a left and a right ideal, then $A$ is a two-sided ideal, or simply an ideal of $M$.

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DEFINITION 3 ([2]). An ideal $P$ of $M$ is said to be prime if for every ideals $A, B$ of $M$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

PROPOSITION 1 ([2]). Let $P$ be an ideal of $M$. Then the following are equivalent:
(a) $P$ is a prime ideal of $M$.
(b) For all $x, y \in M$, $xM y \subseteq P$ implies $x \in P$ or $y \in P$.

PROPOSITION 2 ([3]). Let $I$ be an ideal of $M$. If $P$ is a prime ideal of $M$, then $P \cap I$ is a prime ideal of $I$.

DEFINITION 4 ([1]). A mapping $\theta : M \rightarrow M'$ is called a $\Gamma$-homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x \alpha y) = \theta(x) \alpha \theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

DEFINITION 5 ([12]). Let $\theta : M \rightarrow M'$ be any function and let $\mu$ be any fuzzy set in $M$. The fuzzy set $\eta$ in $M'$ defined by

$$
\eta(y) = \begin{cases} 
\sup_{x \in \theta^{-1}(y)} \mu(x) & \text{if } \theta^{-1}(y) \neq \emptyset, y \in M', \\
0 & \text{otherwise},
\end{cases}
$$

is called the image of $\mu$ under $\theta$, denoted by $\theta(\mu)$.

DEFINITION 6 ([6]). A fuzzy set $\mu$ in $M$ is called a fuzzy left (right) ideal of $M$ if
(4) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(5) $\mu(x \alpha y) \geq \mu(y)$ \quad ($\mu(x \alpha y) \geq \mu(x)$),
for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set $\mu$ in $M$ is called a fuzzy ideal of $M$ if $\mu$ is both a fuzzy left and a fuzzy right ideal of $M$.

We note that $\mu$ is a fuzzy ideal of $M$ if and only if
(4) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(6) $\mu(x \alpha y) \geq \max\{\mu(x), \mu(y)\}$,
for all $x, y \in M$ and all $\alpha \in \Gamma$.

THEOREM 1. Let $\theta : M \rightarrow M'$ be an onto $\Gamma$-homomorphism. If $\mu$ is a fuzzy ideal of $M$, then $\theta(\mu)$ is a fuzzy ideal of $M'$.
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**Proof.** Let \( x', y' \in M' \) and \( \alpha \in \Gamma \). Then there exist \( x, y \in M \) such that \( \theta(x) = x' \) and \( \theta(y) = y' \). Then

\[
\theta(\mu)(x' - y') = \sup_{\theta(z) = z' - y'} \mu(z) \\
\geq \sup_{\theta(x) = x', \theta(y) = y'} \mu(x - y) \\
\geq \sup_{\theta(x) = x'} \sup_{\theta(y) = y'} \min\{\mu(x), \mu(y)\} \\
= \min\{\sup_{\theta(x) = x'} \mu(x), \sup_{\theta(y) = y'} \mu(y)\} \\
= \min\{\theta(\mu)(x'), \theta(\mu)(y')\}
\]

and

\[
\theta(\mu)(x' \alpha y') = \sup_{\theta(z) = x' \alpha y'} \mu(z) \\
\geq \sup_{\theta(x) = z', \theta(y) = y'} \mu(x \alpha y) \\
\geq \sup_{\theta(x) = z'} \max\{\mu(x), \mu(y)\} \\
= \max\{\sup_{\theta(x) = x'} \mu(x), \sup_{\theta(y) = y'} \mu(y)\} \\
= \max\{\theta(\mu)(x'), \theta(\mu)(y')\}
\]

This completes the proof.

**Definition 7 ([5]).** Let \( \mu \) and \( \nu \) be fuzzy sets in \( M \) and let \( \alpha \in \Gamma \). The product \( \mu \Gamma \nu \) is defined by \( \mu \Gamma \nu(x) = \sup_{x = \gamma \alpha \gamma} \min\{\mu(y), \nu(z)\} \) and \( \mu \Gamma \nu(x) = 0 \) if \( x \) is not expressible as \( x = y \alpha z \).

**Definition 8 ([5]).** A fuzzy ideal \( \mu \) of \( M \) is said to be prime if

1. \( \mu \) is not a constant function,
2. for any fuzzy ideals \( \nu, \rho \) in \( M \), \( \nu \Gamma \rho \subseteq \mu \) implies \( \nu \subseteq \mu \) or \( \rho \subseteq \mu \).

**Lemma 1 ([5]).** If \( \mu \) is any nonconstant fuzzy set in \( M \), then \( \mu \) is a fuzzy prime ideal of \( M \) if and only if \( \text{Im}(\mu) = \{1, t\} \) where \( t \in [0, 1] \) and the ideal \( M_\mu = \{x \in M | \mu(x) = 1\} \) is prime.
THEOREM 2. Let \( \mu \) be a fuzzy ideal of \( M \) such that \( 1 \in \text{Im}(\mu) \) and let \( \nu \) be a fuzzy prime ideal of \( M \). Then \( \mu \cap \nu \) is a fuzzy prime ideal of the \( \Gamma \)-ring \( M_\mu = \{ x \in M | \mu(x) = 1 \} \).

Proof. Since \( \nu \) is a fuzzy prime ideal of \( M \), it follows from Lemma 1 that there exists \( t \in [0,1) \) such that

\[
\nu(x) = \begin{cases} 
1 & \text{if } x \in M_\nu, \\
t & \text{otherwise},
\end{cases}
\]

where \( M_\nu = \{ x \in M | \nu(x) = 1 \} \). As \( M_\nu \) is a prime ideal of \( M \), \( M_\mu \cap M_\nu \) is a prime ideal of \( M_\mu \). Now

\[
(\mu \cap \nu)(x) = \begin{cases} 
1 & \text{if } x \in M_\mu \cap M_\nu, \\
t & \text{if } x \in M_\mu - (M_\mu \cap M_\nu).
\end{cases}
\]

Consequently \( \mu \cap \nu \) is a fuzzy prime ideal of \( M_\mu \).

THEOREM 3. Let \( \mu \) be any fuzzy ideal of \( M \). If \( \text{Im}(\mu) = \{ t_0, t_1, ..., t_m \} \) where \( t_0 > t_1 > ... > t_m \) and each \( \mu_i \) is a prime ideal of \( M \), then \( \mu(xa_\beta z) = \max\{ \mu(x), \mu(y), \mu(z) \} \) for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

Proof. Let \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). Without loss of generality, we may assume that \( \max\{ \mu(x), \mu(y), \mu(z) \} = \mu(z) = t_i, 0 \leq i \leq m \). Since \( \mu \) is a fuzzy ideal of \( M \), it follows that

\[
\mu(xa_\beta z) \geq \max\{ \mu(xa_\beta y), \mu(z) \}
\]

\[
\geq \max\{ \mu(x), \mu(y), \mu(z) \}
\]

\[= t_i.\]

Suppose that \( \mu(xa_\beta z) > t_i \). Then \( \mu(xa_\beta z) \in \{ t_0, t_1, ..., t_{i-1} \} \), and hence \( xa_\beta z \in \mu_{t_{i-1}} \). As \( \mu_{t_{i-1}} \) is a prime ideal, it follows from Proposition 1 that \( x \in \mu_{t_{i-1}} \) or \( z \in \mu_{t_{i-1}} \); i.e., \( \mu(x) \geq t_{i-1} \) or \( t_i = \mu(z) \geq t_{i-1} \). This is a contradiction and the proof is complete.

THEOREM 4. Let \( \mu_i \) be any chain of fuzzy prime ideals of \( M \). Then \( \bigcup \mu_i \) is a fuzzy prime ideal of \( M \).

Proof. It is easily proved that \( \bigcup \mu_i \) is a fuzzy ideal of \( M \). Since each \( \mu_i \) is prime, it follows from Lemma 1 that for all \( i \),
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(i) \( 1 \in \text{Im}(\mu_i) \),

(ii) the ideal \( M_{\mu_i} = \{ x \in M | \mu_i(x) = 1 \} \) is prime,

(iii) there exists \( t_i \in (0, 1) \) such that \( \mu_i(x) = t_i \) for all \( x \in M - M_{\mu_i} \). As \( \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_n \subseteq \cdots \) (say), we have \( M_{\mu_1} \subseteq M_{\mu_2} \subseteq \cdots \subseteq M_{\mu_n} \subseteq \cdots \), and hence \( \cup M_{\mu_i} \) is a prime ideal of \( M \). Now let \( \sigma \) and \( \rho \) be any two fuzzy ideals of \( M \) such that \( \sigma \Gamma \rho \subseteq \cup \mu_i \). Assume that \( \cup \mu_i \) is not fuzzy prime. Then there exist \( x, y \in M \) such that \( \sigma(x) > (\cup \mu_i)(x) \) and \( \rho(y) > (\cup \mu_i)(y) \). Therefore \( (\cup \mu_i)(x) \neq 1 \) and \( (\cup \mu_i)(y) \neq 1 \), so that \( x, y \notin \cup M_{\mu_i} \). Since \( \cup M_{\mu_i} \) is prime, it follows from Proposition 1 that \( x \Gamma M y \notin \cup M_{\mu_i} \). Hence \( (\cup \mu_i)(x) = (\cup \mu_i)(y) = (\cup \mu_i)(x \alpha z \beta y) = \text{sup} \), for all \( z \in M \) and \( \alpha, \beta \in \Gamma \). Then

\[
(\sigma \Gamma \rho)(x \alpha z \beta y) \geq \min\{\sigma(x), \rho(y)\} \\
> \min\{(\cup \mu_i)(x), (\cup \mu_i)(y)\} \\
= (\cup \mu_i)(x \alpha z \beta y)
\]

for all \( z \in M \) and \( \alpha, \beta \in \Gamma \). This is a contradiction, and the proof is complete.

**Definition 9** ([12]). Let \( \theta : M \to M' \) be any function. A fuzzy set \( \mu \) in \( M \) is called \( \theta \)-invariant if \( \theta(x) = \theta(y) \) implies \( \mu(x) = \mu(y) \), where \( x, y \in M \).

**Lemma 2.** Let \( \theta : M \to M' \) be an onto \( \Gamma \)-homomorphism and let \( \mu \) be any \( \theta \)-invariant fuzzy ideal of \( M \) such that \( \text{Im}(\mu) = \{ t_0, t_1, \ldots, t_m \} \) where \( t_0 > t_1 > \cdots > t_m \). If the chain of level ideals of \( \mu \) is \( \mu_{t_0} \subset \mu_{t_1} \subset \cdots \subset \mu_{t_m} = M \), then the chain of level ideals of \( \theta(\mu) \) is given by \( \theta(\mu_{t_0}) \subset \theta(\mu_{t_1}) \subset \cdots \subset \theta(\mu_{t_m}) = M' \).

**Proof.** Clearly \( \text{Im}(\theta(\mu)) \subseteq \text{Im}(\mu) \). If \( (\theta(\mu))_{t_i} = \theta(\mu_{t_i}) \), then the chain of level ideals of \( \theta(\mu) \) is given by \( \theta(\mu_{t_0}) \subseteq \theta(\mu_{t_1}) \subseteq \cdots \subseteq \theta(\mu_{t_m}) = M' \). Hence we need only to prove that \( (\theta(\mu))_{t_i} = \theta(\mu_{t_i}) \). Let \( y \in (\theta(\mu))_{t_i} \). Then \( t_i \leq \theta(\mu)(y) = \sup_{z \in \theta^{-1}(y)} \mu(z) \), and so \( \mu(x) \geq t_i \) for some \( x \in \theta^{-1}(y) \). Hence \( x \in \mu_{t_i} \), and \( y = \theta(x) \in \theta(\mu_{t_i}) \), showing that \( (\theta(\mu))_{t_i} \subseteq \theta(\mu_{t_i}) \). To prove the reverse inclusion, let \( y \in \theta(\mu_{t_i}) \). Then there exists \( x \in \mu_{t_i} \), such that \( y = \theta(x) \). Thus \( \mu(x) \geq t_i \), and \( \theta(\mu)(y) = \sup_{z \in \theta^{-1}(y)} \mu(z) \geq \mu(x) \geq t_i \), which means that \( y \in (\theta(\mu))_{t_i} \). This completes the proof.
THEOREM 5. Let $\theta : M \rightarrow M'$ be an onto $\Gamma$-homomorphism. If $\mu$ is a $\theta$-invariant fuzzy prime ideal of $M$, then $\theta(\mu)$ is a fuzzy prime ideal of $M'$.

Proof. Since $\mu$ is a fuzzy prime ideal of $M$, it follows from Lemma 1 that (i) $1 \in \text{Im}(\mu)$, (ii) the ideal $M_\mu = \{x \in M | \mu(x) = 1\}$ is prime and (iii) there exists $t \in [0, 1)$ such that $\mu(x) = t$ for all $x \in M - M_\mu$. Now we show that $\ker(\theta) \subseteq M_\mu$. Let $x \in \ker(\theta)$. Then $\theta(x) = 0_{M'} = \theta(0_M)$. As $\mu$ is $\theta$-invariant and $\mu(0_M) = 1$, we have that $\mu(x) = \mu(0_M) = 1$. Hence $x \in M_\mu$, showing that $\ker(\theta) \subseteq M_\mu$. In view of the fact that $M_\mu$ is a prime ideal and $\ker(\theta) \subseteq M_\mu$, we have that $\theta(M_\mu)$ is a prime ideal of $M'$. Since $M_\mu \subset M$, it follows from Lemma 2 that $\theta(M_\mu) \subseteq \theta(M) = M'$ (the inclusion is strict, because $\mu$ is $\theta$-invariant). Now $\theta(\mu)(0_{M'}) = \sup_{x \in \theta^{-1}(0_{M'})} \mu(x) \geq \mu(0_M) = 1$, and hence $1 \in \text{Im}(\theta(\mu))$. Therefore the result follows from Lemma 1.

References

5. Y. B. Jun, On fuzzy prime ideals of $\Gamma$-rings, (submitted)
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