PROPERTY \((P)\) ON \(\ell_p\)

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\(X\) and \(Y\) are real Banach spaces with closed unit balls \(B_X\) and \(B_Y\) respectively. For \(n = 0, 1, \ldots\), the space \(\mathcal{P}^{(n)}(X, Y)\) of continuous \(n\)-homogeneous polynomials \(P : X \to Y\) consists of all functions \(P\) of the form \(P(x) = A(x, \ldots, x)\), where \(A : X \times \cdots \times X \to Y\) is a continuous \(n\)-linear mapping. \(\|P\| = \sup\{|P(x)| : x \in B_X\}\). The space \(P(X, Y)\) is the algebraic direct sum of the space \(\mathcal{P}^{(n)}(X, Y), n = 0, 1, 2, \ldots\) \(\mathcal{P}(X)\) and \(\mathcal{P}(Y)\) denote \(\mathcal{P}(X, \mathbb{R})\) and \(\mathcal{P}(Y, \mathbb{R})\), respectively.

We say that a Banach space \(X\) has property \((P)\) if for any bounded sequences \((u_n)\) and \((v_n)\) in \(X\) such that \(|P(u_n) - P(v_n)| \to 0\) as \(n \to \infty\) for all \(P \in \mathcal{P}^{(n)}(X)\), \(n \geq 1\), then \(|Q(u_n) - Q(v_n)| \to 0\) as \(n \to \infty\) for all \(Q \in \mathcal{P}^{(n)}(X)\), \(n \geq 1\). This property was studied in [ACL], which is closely related with Dunford-Pettis property and Schur property. Aron, Choi and Llavona [ACL] showed that every super-reflexive Banach space has property \((P)\). However in their proof they used the fact that every super-reflexive Banach space is in \(W_p\)-class, which was studied by Castillo and Sánchez [CS]. Hence we cannot have the exact form of polynomial which works in their proof. In this note we will prove that every \(\ell_p\) \((1 < p < \infty)\) has property \((P)\), without using \(W_p\)-class. For general background on polynomials we refer to [D] and [M].

**Lemma 1.** For any \(p\), \(1 \leq p < \infty\), if \((u_j)\) and \((v_j)\) are two sequences in \(\ell_p\) which go to 0 weakly, and if for all polynomials \(P \in \mathcal{P}(\ell_p)\), \(P(u_j) - P(v_j) \to 0\), then \(\|u_j - v_j\|_p \to 0\).

**Proof.** It is enough to prove the case \(1 < p < \infty\). Suppose that \(\|u_j - v_j\|_p \neq 0\). Since \((u_j)\) and \((v_j)\) converge to 0 weakly, by passing to a subsequence, there is an increasing sequence \((n_j)\) of positive integers such that for \(E_j = \{n_j \leq k \leq n_{j+1} - 1\}\),

\[
\|u_j x E_j - u_j\|_p < \frac{1}{2^j} \quad \text{and} \quad \|v_j x E_j - v_j\|_p < \frac{1}{2^j}.
\]

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We may assume \((u_j)\) and \((v_j)\) to be \((u_j \chi_{E_j})\) and \((v_j \chi_{E_j})\) respectively. Let

\[
E_j^+ = \{ k \in E_j : u_j^k v_j^k \geq 0 \}, \quad \text{and} \quad E_j^- = \{ k \in E_j : u_j^k v_j^k < 0 \},
\]

where \(u_j = (u_j^1, u_j^2, \ldots, u_j^n, \ldots)\) and \(v_j = (v_j^1, v_j^2, \ldots, v_j^n, \ldots)\). Since \(\|u_j - v_j\|_p \neq 0\). We have that \(\| (u_j - v_j) \chi_{E_j^+} \|_p \neq 0\) or \(\| (u_j - v_j) \chi_{E_j^-} \|_p \neq 0\). When \(\| (u_j - v_j) \chi_{E_j^+} \|_p \neq 0\), we partition each set \(E_j^+\) into four pairwise disjoint subsets. \(P_{u_j}, P_{v_j}, N_{u_j}, N_{v_j}\) where

\[
\begin{align*}
P_{u_j} &= \{ k \in E_j^+ : u_j^k \geq v_j^k > 0 \}, \\
P_{v_j} &= \{ k \in E_j^+ : v_j^k > u_j^k > 0 \}, \\
N_{u_j} &= \{ k \in E_j^+ : u_j^k \leq v_j^k \leq 0 \} \quad \text{and} \\
N_{v_j} &= \{ k \in E_j^+ : v_j^k < u_j^k \leq 0 \}.
\end{align*}
\]

Since \(\| (u_j - v_j) \chi_{E_j^+} \|_p \neq 0\), one of the following four sequences does not converge to 0: \(\| (u_j - v_j) \chi_{P_{u_j}} \|_p \), \(\| (u_j - v_j) \chi_{P_{v_j}} \|_p \), \(\| (u_j - v_j) \chi_{N_{u_j}} \|_p \), \(\| (u_j - v_j) \chi_{N_{v_j}} \|_p \). Suppose \(\| (u_j - v_j) \chi_{P_{u_j}} \|_p \) does not converge to 0. We may assume that for each \(j\), \(\| (u_j - v_j) \chi_{P_{u_j}} \|_p \geq \delta\) for some \(\delta > 0\) (consider a subsequence if necessary). Define

\[
P(x) = \sum_{j=1}^{\infty} \left( \sum_{k \in P_{u_j}} x^k |u_j^k - v_j^k|^{p-1} \right)^N
\]

where \(N\) is an integer greater than \(p\) and \(x = (x^1, x^2, \ldots, x^n, \ldots) \in \ell_p\). If \(\|x\|_p \leq 1\), then

\[
|P(x)| \leq \sum_{j=1}^{\infty} \left( \sum_{k \in P_{u_j}} |x^k||u_j^k - v_j^k|^{p-1} \right)^N
\leq \sum_{j=1}^{\infty} \left( \sum_{k \in P_{u_j}} |x^k|^p \right)^{\frac{N}{p}} \left( \sum_{k \in P_{u_j}} |u_j^k - v_j^k|^{(p-1)q} \right)^{\frac{N}{q}}
\]

\((q\) is the exponential conjugate of \(p\)).
Since \((u_j)\) and \((v_j)\) are bounded sequences in \(\ell_p\), there exists a constant \(C > 0\) such that
\[
\left( \sum_{k \in P_{u_j}} |u_j^k - v_j^k|^{(p-1)q} \right)^{\frac{N}{q}} = \left( \sum_{k \in P_{u_j}} |u_j^k - v_j^k|^p \right)^{\frac{N}{q}} \leq C
\]
for every \(j\). Thus
\[
|P(x)| \leq C \sum_{j=1}^{\infty} \left( \sum_{k \in P_{u_j}} |x^k|^p \right)^{\frac{N}{p}} \leq C \|x\|_p^p \leq C,
\]
for \(\|x\|_p \leq 1\). The second inequality above comes from the fact \(\frac{N}{p} \geq 1\) and \(\|x\|_p \leq 1\). Thus \(P\) is a continuous \(N\)-homogeneous polynomial on \(\ell_p\). However
\[
P(u_\ell) - P(v_\ell) = \left( \sum_{k \in P_{u_\ell}} u_\ell^k |u_\ell^k - v_\ell^k|^{p-1} \right)^{\frac{N}{p}} - \left( \sum_{k \in P_{v_\ell}} v_\ell^k |u_\ell^k - v_\ell^k|^{p-1} \right)^{\frac{N}{p}} \geq \left( \sum_{k \in P_{u_\ell}} |u_\ell^k - v_\ell^k|^p \right)^{\frac{N}{p}} = \|(u_\ell - v_\ell)\chi_{P_{u_\ell}}\|_p^p \geq \delta \|P\|_p^N,
\]
which contradicts hypothesis. (The first inequality above from the fact that if \(a, b > 0\), then \((a - b)^N \geq a^N - b^N\).) The other cases are proved similarly to the above.

On the other hand, when \(\|(u_j - v_j)\chi_{E_j^-}\|_p \not\to 0\), we have that \(\|u_j\chi_{E_j^-}\|_p \not\to 0\) or \(\|v_j\chi_{E_j^-}\|_p \not\to 0\). We may assume without loss of generality that for each \(j\), \(\|u_j\chi_{E_j^-}\|_p \geq \delta\) for some \(\delta > 0\) (consider a subsequence if necessary). Let \(z_j = \frac{u_j\chi_{E_j^-}}{\|u_j\chi_{E_j^-}\|_p}\) and then \((z_j)\) is a normalized basic sequence in \(\ell_p\). Let \(Z\) be the closed subspace of \(\ell_p\).
spanned by \((z_j)\). Define \(\pi : \ell_p \to \ell_p\) by

\[
\pi(x) = \sum_{j=1}^{\infty} \left( \sum_{k \in E_j^-} x^k (\text{sign } z_j^k)|z_j^k|^{p-1} \right) z_j
\]

\((x = (x^1, x^2, \ldots) \in \ell_p)\).

Since \((z_j)\) is a normalized sequence with pairwise disjoint supports,

\[
\|\pi(x)\|_p = \left( \sum_{j=1}^{\infty} \left| \sum_{k \in E_j^-} x^k (\text{sign } z_j^k)|z_j^k|^{p-1} \right|^p \right)^{\frac{1}{p}}
\]

\[
\leq \left[ \sum_{j=1}^{\infty} \left( \sum_{k \in E_j^-} |x^k|^p \right)^{\frac{p}{q}} \left( \sum_{k \in E_j^-} |z_j^k|^{(p-1)q} \right) \right]^{\frac{q}{p}}
\]

Since \(\sum_{k \in E_j^-} |z_j^k|^{(p-1)q} = \sum_{k \in E_j^-} |z_j^k|^p = 1\) for every \(j\), we obtain \(\|\pi(x)\|_p \leq \|x\|_p\) and also \(\pi(z_j) = z_j\) for every \(j\). Hence \(\pi\) is a norm 1 projection from \(\ell_p\) onto \(Z\).

Define \(P : Z \to \mathbb{R}\) by \(P(\sum a_j z_j) = \sum a_j z_j^N\) where \(N\) is an odd integer greater than \(p\). It is easy to see \(P \in \mathcal{P}(N\mathbb{Z})\) and hence \(\tilde{P} = P \circ \pi \in \mathcal{P}(N\ell_p)\). For each \(\ell\), we get

\[
\tilde{P}(u_\ell) = (P \circ \pi)(u_\ell)
\]

\[
= \left( \sum_{k \in E_\ell^-} u_\ell^k (\text{sign } z_\ell^k)|z_\ell^k|^{p-1} \right)^N
\]

\[
= \left( \|u_\ell \chi_{E_\ell^-}\|_p \sum_{k \in E_\ell^-} |z_\ell^k|^p \right)^N
\]

\(= \|u_\ell \chi_{E_\ell^-}\|_p^N \geq \delta^N\)
and
\[ \tilde{P}(v_\ell) = \left( \sum_{k \in \mathcal{E}_*} v_k^{(\text{sign } z_k^{(\ell)})}|z_k^{(\ell)}|^{p-1} \right)^N < 0. \]

This implies \( \tilde{P}(u_\ell) - \tilde{P}(v_\ell) \geq \delta^N \) for every \( \ell \), which contradicts hypothesis. \( \square \)

**Theorem 2.** For any \( 1 \leq p \leq \infty \), \( \ell_p \) has property (P).

**Proof.** We only need to consider \( p \in (1, \infty) \), since \( \ell_1 \) and \( \ell_\infty \) have the Dunford Pettis property. Let \( (u_n) \) and \( (v_n) \) be bounded sequences in \( X \) such that \( |P(u_n) - P(v_n)| \to 0 \) for all polynomials \( P \). Using the reflexivity of \( \ell_p \), we may suppose without loss of generality that both \( (u_n) \) and \( (v_n) \) tend weakly to some \( x \in \ell_p \). Moreover, our hypothesis implies that for all continuous polynomials \( P \), \( P(u_n - x) - P(v_n - x) \to 0 \). To see this, let \( A \) be the continuous symmetric \( k \)-linear form associated with a continuous \( k \)-homogeneous polynomial \( P_k \). Thus

\[
P_k(u_n - x) - P_k(v_n - x) = A(u_n - x, \ldots, u_n - x) - A(v_n - x, \ldots, v_n - x)
\]

\[
= \sum_{i=1}^k \left[ A(u_n, \ldots, u_n) - A(v_n, \ldots, v_n) \right] + k \left[ A(x, u_n, \ldots, u_n) - A(x, v_n, \ldots, v_n) \right] \]

\[
+ \cdots + k \left[ A(x, \ldots, x, u_n) - A(x, \ldots, x, v_n) \right] \]

\[
= P_k(u_n) - P_k(v_n) + k [P_{k-1}(u_n) - P_{k-1}(v_n)]
\]

\[
+ \cdots + k [P_1(u_n) - P_1(v_n)],
\]

where \( P_i \) is the \( i \)-homogeneous polynomial defined by \( P_i(y) = A(x^{k-i}y^i) \), \( (1 \leq i \leq k) \).

Therefore, the sequences \( (u_n - x) \) and \( (v_n - x) \) satisfy the conditions of the preceding lemma. Hence \( \|(u_n - x) - (v_n - x)\| = \|u_n - v_n\| \to 0 \). Thus, for every \( Q \in P^n(X), n \geq 1 \), \( Q(u_n - v_n) \to 0 \). \( \square \)

**References**


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