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COLLOCATION APPROXIMATIONS FOR INTEGRO–DIFFERENTIAL EQUATIONS

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1. Introduction

This paper concerns collocation methods for integro-differential equations in which memory kernels have a singularity at $t = 0$. There has been extensive research in recent years on Volterra integral and integro-differential equations for physical systems with memory effects in which the stability and asymtotic stability of solutions have been the main interest. We will study a class of hereditary equations with singular kernels which interpolate between well known model equations as the order of singularity varies. We are also concerned with the smoothing effect of singular kernels, but we use energy methods and our results involve fractional time in fixed spatial norms. Galerkin methods for our models was studied and existence, uniqueness and stability results was obtained in [4]. Our major goal is to study collocation methods.

Let Δu have its domain

$$H^2(0,1) \cap H_0^1(0,1) \qquad (u(0) = u(1) = 0),$$

where Δ is Laplacian, $H^2(0,1)$ and $H_0^1(0,1)$ are the usual Sobolev spaces. See [7]. Set

$$a_0(t) = e^{-t}, \qquad t > 0$$

and define a linear hereditary operator L_{a_0} by

$$L_{a_0}[u](t) = \int_0^t a_0(t-z)u(z)dz, \qquad t > 0. \tag{1.1}$$

Consider the following equations:

(P) $\qquad u_t - \Delta u = f, \qquad$ parabolic heat flow model.

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(GP)

$$u_t - L_{a_0}[\triangle u] = f,$$ Gurtin -Pipkin heat flow model in materials with memory [5].

(V) $$u_{tt} - 2\triangle u + L_{a_0}[\triangle u] = f,$$ viscoelasticity model [9].

(K) $$u_{tt} - \triangle u - \triangle u_t = f,$$ Kelvin solid model [7].

Let, for $0 < \alpha < 1$,

$$a_\alpha(t) = \frac{e^{-t}}{\Gamma(1-\alpha)t^\alpha},$$

where $\Gamma(1-\alpha)$ is the gamma function. The fractional derivative is defined by

$$D^\alpha \phi(t) = \frac{d}{dt}\int_0^t \frac{\phi(z)}{\Gamma(1-\alpha)(t-z)^\alpha}dz, \qquad 0 < \alpha < 1. \tag{1.2}$$

Observe that the right hand side of (1.2) tends to $D\phi(t)$ as $\alpha \to 1$. Thus we write formally

$$D^0\phi(t) = \phi(t) \qquad : \qquad D^1\phi(t) = D\phi(t) \qquad [1],[6].$$

We see that the Laplace transform $\mathcal{L}[\cdot]$ of the fractional derivative is

$$\mathcal{L}[D^\alpha \phi](s) = s^\alpha \hat{\phi}(s), \qquad 0 < \alpha < 1.$$

We now define the " singular interpolation" hereditary operator L_{a_α} by

$$L_{a_\alpha}[u](t) = \int_0^t a_\alpha(t-z)u(z)dz, \qquad t > 0. \tag{1.3}$$

It follows then that

$$L_{a_\alpha}[u](t) \to L_{a_0}[u](t) \qquad as \qquad \alpha \to 0$$

and

$$L_{a_\alpha}[u](t) \to u(t) \qquad as \qquad \alpha \to 1.$$

In this sense $L_{a_\alpha}[\cdot]$ is an interpolation between $L_{a_0}[\cdot]$ and the identity which we term "singular interpolation".

We now introduce our models (GPP) and (VK):

$$\text{(GPP)} \qquad u_t - \int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta u(z)dz = f : \qquad u(0) = u_0.$$

(VK)

$$u_{tt} - \Delta u - \frac{d}{dt}\int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta u(z)dz = f :$$
$$u_t(0) = u_1, u(0) = u_0.$$

(GPP) and (VK) are interpolation between (GP) and (P), (V) and (K) respectively. We will restrict ourselves throughout to the cases in which the initial values are zero.

2. Collocation Approximations

2-1. Stability for collocation methods

Let $\Omega = (0,1) \times (0,1)$ and for any positive integer N, we set $P_N = {}_1P_N \times {}_1P_N$, where ${}_1P_N$ is the space of the polynomials of degree N in a single variable. Further we set

$$P_N^0 = \{p \in P_N : p(x) = 0 \text{ if } x \in \partial\Omega\}, \partial\Omega = \text{boundary of } \Omega.$$

Let $\{x_{ij}\}_{i,j=0}^N$ be the points in $\bar{\Omega}$ such that $x_{ij} = (x_i, y_j)$ where for fixed $N > 1$, set $h = \frac{1}{N}$ and

$$x_i = ih, y_j = jh, \qquad \forall \quad 0 \le i,j \le N.$$

that is, points in the uniform partition of $\bar{\Omega} = [0,1] \times [0,1]$. Define

$$\text{(2.1.1)} \qquad \int_0^1 p(x)dx = \sum_{i=0}^N p(x_i)w_i \qquad \text{for all } p \in {}_1P_N,$$

where w_i's are suitable weights such that the numerical quadrature holds exactly for all polynomials with degree up to N.

We define a discrete inner product

$$(2.1.2)\qquad (\phi,\psi)_N = \sum_{i,j=0}^{N} \phi(x_{ij})\psi(x_{ij})w_i w_j,$$
$$\text{for all } \phi,\psi \in C^0(\bar{\Omega}), \bar{\Omega} = \Omega \cup \partial\Omega.$$

By (2.1.1) we have

$$(2.1.3)\qquad (\phi,\psi)_N = (\phi,\psi) \qquad \text{for all } \phi,\psi : \phi\cdot\psi \in P_N.$$

Let $\tilde{P}_N$ be the complexification of P_N. Then (2.1.1) and (2.1.2) can be extended as following :

$$(\widetilde{2.1.1})\qquad \int_0^1 \tilde{p}(x)dx = \sum \tilde{p}_R(x)w_j + i\sum \tilde{p}_I(x)w_j,$$

where

$$\tilde{p}_R = \text{real part of } \tilde{p},$$
$$\tilde{p}_I = \text{ imaginary part of } \tilde{p},$$

and

$(\widetilde{2.1.2})$

$$(\tilde{\phi},\tilde{\psi})_N = \sum_{i,j=0}^{N} \tilde{\phi}(x_{ij})\overline{\tilde{\psi}(x_{ij})}w_i w_j$$
$$= \sum(\tilde{\phi}_R\tilde{\psi}_R + \tilde{\phi}_I\tilde{\psi}_I)w_i w_j + i\sum(\tilde{\phi}_I\tilde{\psi}_R - \tilde{\phi}_R\tilde{\psi}_I)w_i w_j,$$
$$\text{for all } \tilde{\phi},\tilde{\psi} : \tilde{\phi}\cdot\tilde{\psi} \in \tilde{P}_N,$$
$$\text{for all} \tilde{\phi},\tilde{\psi} \in \widetilde{C^0(\bar{\Omega})} = \text{ complexification of } C^0(\bar{\Omega})$$

Then by $(\widetilde{2.1.1})$,

$$(\widetilde{2.1.3})\qquad (\tilde{\phi},\tilde{\psi})_N = (\tilde{\phi},\tilde{\psi}) \qquad \text{for all } \tilde{\phi},\tilde{\psi} : \tilde{\phi}\cdot\tilde{\psi} \in \tilde{P}_N.$$

We define the interpolation operators I_N and $\tilde{I}_N$ by

$$I_N : C^0(\bar{\Omega}) \to P_N : I_N v(x_{ij}) = v(x_{ij}), \quad v \in C^0(\bar{\Omega}), 0 \le i,j \le N$$

and

$$\tilde{I}_N : \widetilde{C^0(\bar{\Omega})} \to \tilde{P}_N : \tilde{I}_N \tilde{v}(x_{ij}) = \tilde{v}(x_{ij}), \quad \tilde{v} \in \widetilde{C^0(\bar{\Omega})}, 0 \le i,j \le N.$$

For real $\sigma > 0$, we define H^σ by interpolation between $H^{[\sigma]}$ and $H^{[\sigma+1]}$, where $[\sigma]$ denotes the integer part of σ.

For $v \in H^\sigma(-1,1)$ with $\sigma \ge 1$, the interpolation error can be estimate as follows (see [3]) :

(2.1.4) $$\| v - I_N v \|_0 \le C h^\sigma \| v \|_\sigma, \qquad \text{for all } v \in H^\sigma$$

and

$(\widetilde{2.1.4})$ $$\| \tilde{v} - \tilde{I}_N \tilde{v} \|_0 \le C h^\sigma \| \tilde{v} \|_\sigma, \qquad \text{for all } \tilde{v} \in \tilde{H}^\sigma.$$

Note that

$$(\phi, \psi)_N = (I_N \phi, \psi)_N, \qquad \text{for all } \phi, \psi \in C^0(\bar{\Omega})$$

and

$$(\tilde{\phi}, \tilde{\psi})_N = (\tilde{I}_N \tilde{\phi}, \tilde{\psi})_N, \qquad \text{for all } \tilde{\phi}, \tilde{\psi} \in \widetilde{C^0(\bar{\Omega})}$$

The discrete norm

$$\| \phi \|_N = \{(\phi, \phi)_N\}^{\frac{1}{2}}, \qquad \phi \in C^0(\bar{\Omega})$$

is equvalent to the L^2 - norm,

(2.1.5) $$\| \phi \| \le \| \phi \|_N \le 2 \| \phi \|, \qquad \phi \in C^0(\bar{\Omega})$$

(see [2]).

We formulate collocation methods for the problems (GPP) and (VK) and investigate existence, uniqueness and stability results with collocation methods. For the existence and uniqueness of solutions for (GPP) and (VK), we refer to [4].

The collocation method has the advantage that it is usually very easy to implement, even when the problem to be solved is highly nonlinear. For simplicity we ass· ıe that all initial conditions are zeros.

The semidiscrete pseudospectral approximations of (GPP) and (VK) consist in the following collocation problems : we look for mapping $U^h \in C^2(P_N^0)$ such that for any $t \in (0,T)$,

(GPPC)

$$U_t^h(x_{ij},t) - \int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta U^h(x_{ij},z)dz = f(x_{ij},t),$$
$$1 \le i,j \le N-1,$$
$$U^h(x_{ij},0) = 0, \qquad 0 \le i,j \le N.$$

(VKC)

$$U_{tt}^h(x_{ij},t) - \Delta U^h(x_{ij},t) - \frac{d}{dt}\int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta U^h(x_{ij},z)dz$$
$$= f(x_{ij},t), \qquad 1 \le i,j \le N-1,$$
$$U^h(x_{ij},0) = 0, U_t^h(x_{ij},0) = 0, \qquad 0 \le i,j \le N.$$

If we take the Laplace transform on both sides of (GPPC) and (VKC) then we obtain

$(\widehat{GPPC})$

$$s\hat{U}^h(x_{ij}) - (s+1)^{\alpha-1}\Delta\hat{U}^h(x_{ij}) = \hat{f}(x_{ij}),$$
$$1 \le i,j \le N-1,$$
$$\hat{U}^h(x_{ij}) = 0, \qquad 0 \le i,j \le N.$$

$(\widehat{VKC})$

$$s^2\hat{U}^h(x_{ij}) - \Delta\hat{U}^h(x_{ij}) - s(s+1)^{\alpha-1}\Delta\hat{U}^h(x_{ij}) = \hat{f}(x_{ij}),$$
$$1 \le i,j \le N-1,$$
$$\hat{U}^h(x_{ij}) = 0, s\hat{U}^h(x_{ij}) = 0, \qquad 0 \le i,j \le N.$$

We are now concerned with estimation on $\hat{U}$ for $s = i\eta$. For a function $\hat{u}(s)$ let us define a norm $\| \hat{u} \|_{\gamma,\delta}$ by

$$\| \hat{u}(i\eta) \|^2_{\gamma,\delta} = (1+\eta^2)^\gamma \| \hat{u}(i\eta) \|^2_0 + (1+\eta^2)^\delta \| \hat{u}(i\eta) \|^2_1 .$$

Throughout, $C > 0$ will denote a generic constant.

LEMMA 2.1. *(Stability of $\hat{U}^h$ in $(\widehat{GPPC})$)*
The semidiscrete collocation approximation problems $(\widehat{GPPC})$ are stable, namely

$$(S_1) \qquad \| \hat{U}^h(i\eta) \|_{1,\frac{\alpha}{2}} \leq C_\alpha \| \hat{f}(i\eta) \|_0 .$$

Proof. We recall that

$(\widehat{GPPC})$

$$(s\hat{U}^h(s), \hat{v}^h) + (s+1)^{\alpha-1}(\nabla \hat{U}^h(s), \nabla \hat{v}^h) = (\hat{f}(s), \hat{v}^h)_N,$$
$$\text{for all } \hat{v}^h \in \tilde{P}^0_N.$$

We put $s = i\eta$ and $\hat{v}^h = \hat{U}^h \in \tilde{P}^0_N$ in $(\widehat{GPPC})$ and we have

$$(2.1.6) \qquad i\eta \| \hat{U}^h(i\eta) \|^2_0 + (1+i\eta)^{\alpha-1}(\nabla \hat{U}^h, \nabla \hat{U}^h) = (\hat{f}, \hat{U}^h)_N.$$

(i) For a bounded set $|\eta| \leq K$:
Clearly there exists a constant $\delta_\alpha(K)$ such that, by (2.1.5)
(2.1.7)

$$\| \hat{U}^h(i\eta) \|^2_0 \leq \| \hat{U}^h(i\eta) \|^2_1 \leq \frac{4C}{\delta_\alpha(K)} \| \hat{f}(i\eta) \|_0 \| \hat{U}^h(i\eta) \|_0, \qquad |\eta| \leq K.$$

$$\| \hat{U}^h(i\eta) \|_0 \leq \frac{4C}{\delta_\alpha(K)} \| \hat{f}(i\eta) \|_0 .$$

Note that

$$(1+i\eta)^{\alpha-1} = (1+\eta^2)^{\frac{\alpha-1}{2}}[\cos(\alpha-1)\theta + i\sin(\alpha-1)\theta],$$
$$\theta = arg(1+i\eta).$$

the equation (2.1.6) becomes

(2.1.8)

$$i\eta \parallel \hat{U}^h(i\eta) \parallel_0^2 + (1+\eta^2)^{\frac{\alpha-1}{2}}(\lambda_\alpha + i\mu_\alpha)(\nabla \hat{U}^h, \nabla \hat{U}^h) = (\hat{f}, \hat{U}^h)_N,$$

where $\cos(\alpha-1)\theta = \lambda_\alpha, \sin(\alpha-1)\theta = \mu_\alpha$.

(ii) For large $|\eta| \geq K$:
If we take the real part of (2.1.8), we obtain using (2 1.5)

$$(2.1.9) \qquad (1+\eta^2)^{\frac{\alpha-1}{2}} \parallel \hat{U}^h(i\eta) \parallel_1^2 \leq \frac{C}{\lambda_\alpha} \parallel \hat{f}(i\eta) \parallel_0 \parallel \hat{U}^h(i\eta) \parallel_0 .$$

Taking the imaginary part and using (2.1.9)

$$|\eta| \parallel \hat{U}^h(i\eta) \parallel_0^2 \leq (1+\eta^2)^{\frac{\alpha-1}{2}} \mu_\alpha \parallel \hat{U}^h(i\eta) \parallel_1^2 + 4C \parallel \hat{f}(i\eta) \parallel_0 \parallel \hat{U}^h(i\eta) \parallel_0$$

$$\leq (\frac{C\mu_\alpha}{\lambda_\alpha} + 4) \parallel \hat{f}(i\eta) \parallel_0 \parallel \hat{U}^h(i\eta) \parallel_0 .$$

So it follow that

$$(2.1.10) \qquad |\eta| \parallel \hat{U}^h(i\eta) \parallel_0 \leq C_\alpha \parallel \hat{f}(i\eta) \parallel_0 .$$

$$(2.1.11) \qquad \parallel \hat{U}^h(i\eta) \parallel_0 \leq \frac{C_\alpha}{|\eta|} \parallel \hat{f}(i\eta) \parallel_0 .$$

Thus (2.1.9) and (2.1.11) imply that

$$(1+\eta^2)^{\frac{\alpha-1}{2}} \parallel \hat{U}^h(i\eta) \parallel_1^2 \leq \frac{1}{|\eta|} \frac{C_\alpha}{\lambda_\alpha} \parallel \hat{f}(i\eta) \parallel_0^2 .$$

Hence

$$(2.1.12) \qquad |\eta|^\alpha \parallel \hat{U}^h(i\eta) \parallel_1^2 \leq C_\alpha \parallel \hat{f}(i\eta) \parallel_0^2, \qquad |\eta| \geq K.$$

Thus combining (2.1.7), (2.1.10) and (2.1.12) yields that , for all $\eta \in R$,

$$(2.1.13) \quad (1+\eta^2) \parallel \hat{U}^h(i\eta) \parallel_0^2 + (1+\eta^2)^{\frac{\alpha}{2}} \parallel \hat{U}^h(i\eta) \parallel_1^2 \leq C_\alpha \parallel \hat{f}(i\eta) \parallel_0^2,$$

which implies the assertion

$$\parallel \hat{U}^h(i\eta) \parallel_{1,\frac{\alpha}{2}} \leq C_\alpha \parallel \hat{f}(i\eta) \parallel_0 . \qquad ///$$

LEMMA 2.2. *(Stability of $\hat{U}^h$ in $(\widehat{VKC})$)*
The $(\widehat{VKC})$ are stable, namely

$$(S_2) \qquad \| \hat{U}^h(i\eta) \|_{2,\alpha} \leq C_\alpha \| \hat{f}(i\eta) \|_0 .$$

Proof. We recall that

$(\widehat{VKC})$

$$(s^2\hat{U}^h(s), \hat{v}^h) + (\nabla \hat{U}^h, \nabla \hat{v}^h) + s(s+1)^{\alpha-1}(\nabla \hat{U}^h, \nabla \hat{v}^h) = (\hat{f}(s), \hat{v}^h)_N, \quad \text{for all } \hat{v}^h \in \tilde{P}_N^0.$$

We put $s = i\eta$ and $\hat{v}^h = \hat{U}^h$ in $(\widehat{VKC})$ to obtain

(2.1.14)

$$-\eta^2 \| \hat{U}^h(i\eta) \|_0^2 + (\nabla \hat{U}^h, \nabla \hat{U}^h) + i\eta(1+i\eta)^{\alpha-1}(\nabla \hat{U}^h, \nabla \hat{U}^h) = (\hat{f}, \hat{U}^h)_N.$$

Note that

$$i\eta(1+i\eta)^{\alpha-1} = | \eta | (1+\eta^2)^{\frac{\alpha-1}{2}}[-\sin(\alpha-1)\theta + i\cos(\alpha-1)\theta].$$

We can rewrite (2.1.14) as

(2.1.15)

$$-\eta^2 \| \hat{U}^h(i\eta) \|_0^2 + \| \nabla \hat{U}^h(i\eta) \|_0^2 + | \eta | (1+\eta^2)^{\frac{\alpha-1}{2}}[-\mu_\alpha + i\lambda_\alpha] \times \| \nabla \hat{U}^h(i\eta) \|_0^2 = (\hat{f}, \hat{U}^h)_N.$$

(i) For a bounded set $| \eta | \leq K$:
We have immediately that

(2.1.16)

$$\| \hat{U}^h(i\eta) \|_0^2 \leq \| \hat{U}^h(i\eta) \|_1^2 \leq C(\epsilon, K) \| \hat{f}(i\eta) \|_0 \| \hat{U}^h(i\eta) \|_0, \quad | \eta | \leq K.$$

$$(2.1.17) \qquad \| \hat{U}^h(i\eta) \|_0 \leq C(\epsilon, K) \| \hat{f}(i\eta) \|_0 .$$

(ii) For large $|\eta| \geq K$:
If we take imaginary part of (2.1.15), we obtain

$$(2.1.18) \quad |\eta|^{\alpha} \| \hat{U}^h(i\eta) \|_1^2 \leq \frac{C}{\lambda_\alpha} \| \hat{f}(i\eta) \|_0 \| \hat{U}^h(i\eta) \|_0, \qquad |\eta| \geq K.$$

Next taking the real part yields that

(2.1.19)

$$\begin{aligned} \eta^2 \| \hat{U}^h(i\eta) \|_0^2 \leq & \| \hat{U}^h(i\eta) \|_1^2 + |\eta|^{\alpha} \mu_\alpha \| \hat{U}^h(i\eta) \|_1^2 \\ & + 4C \| \hat{f}(i\eta) \|_0 \| \hat{U}^h(i\eta) \|_0 \\ \leq & C_\alpha \| \hat{f}(i\eta) \|_0 \| \hat{U}^h(i\eta) \|_0 . \end{aligned}$$

Now we combine (2.1.17),(2.1.18) and (2.1.19) to obtain an estimate for all $\eta \in R$,

$$(2.1.20) \quad (1+\eta^2)^2 \| \hat{U}^h(i\eta) \|_0^2 + (1+\eta^2)^{\alpha} \| \hat{U}^h(i\eta) \|_1^2 \leq C_\alpha \| \hat{f}(i\eta) \|_0^2 .$$

From which the result follows.

$$\| \hat{U}^h(i\eta) \|_{2,\alpha} \leq C_\alpha \| \hat{f}(i\eta) \|_0 . \qquad ///$$

Now we state the main Theorems as follows :

THEOREM 2.3. *(Stability of U^h in (GPPC))*
For $0 < \alpha < 1$, there exists a constant c_α such that

$$\| U^h \|_{1,\frac{\alpha}{2}} \leq C_\alpha \| f \|_{L_2(0,\infty:H_0)} .$$

THEOREM 2.4. *(Stability of U^h in (VKC))*
For $0 < \alpha < 1$, there exists a constant C_α such that

$$\| U^h \|_{2,\alpha} \leq C_\alpha \| f \|_{L_2(o,\infty:H_0)} .$$

Proofs. We define a norm $\| U^h \|_{\gamma,\delta}$ by

$$\| U^h \|_{\gamma,\delta} = \int_{-\infty}^{\infty} \| \hat{U}^h(i\eta) \|_{\gamma,\delta}^2 \, d\eta.$$

Then we obtain the assertions by integrating both sides of (S_1) and (S_2) in Lemma 2.1 and 2.2. ///

It follows then that the above two theorems give the followings:

THEOREM 2.5. *(Existence and Uniqueness of U^h of (GPPC) and (VKC))*
(GPPC) and (VKC) have unique solutions respectively.

2-2. Error Analysis

Let (GPPC) and (VKC) can be rewritten :

(GPPC)

$$U_t^h - \int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta U^h dz = f, \qquad h = \frac{1}{N},$$
$$U^h(0) = 0,$$

(VKC)

$$U_{tt}^h - \Delta U^h - \frac{d}{dt}\int_0^t \frac{e^{-(t-z)}}{\Gamma(1-\alpha)(t-z)^\alpha} \Delta U^h dz = f, \ \mathrm{h} = \frac{1}{N}.$$
$$U^h(0) = 0, U_t^h(0) = 0.$$

The cruicial question in collocation methods is whether the approximations U^h converge to the solution. In this section we give error estimates. Let us denote by $\mathcal{H}^h_{\gamma,\delta}$ the space of all functions w^h with values in P_N and with $\| w^h \|_{\gamma,\delta} < \infty$ in the notation of last section. We will establish the following error estimates.

THEOREM 2.6. *There is a constant $C_\alpha > 0$, independent of h, such that if u and U^h are the solutions of (GPP) and (GPPC) respectively, then*

$$\| u - U^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} \leq C_\alpha \Big(\inf_{w^h \in \mathcal{H}^h_{\frac{1}{2},\frac{\alpha-1}{2}}} \| u - w^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} + h^{\sigma-\frac{1}{2}} \| f \|_\sigma \Big).$$

THEOREM 2.7. *There is a constant $C_\alpha > 0$, independent of h, such that if u and U^h are the solutions of (VK) and (VKC) respectively, then*

$$\| u - U^h \|_{1,\frac{\alpha}{2}} \leq C_\alpha \Big(\inf_{w^h \in \mathcal{H}^h_{1,\frac{\alpha}{2}}} \| u - w^h \|_{1,\frac{\alpha}{2}} + h^{\sigma-\frac{1}{2}} \| f \|_\sigma \Big).$$

We will prove the theorems again by going to the transform plane. Recall that

$$(\widehat{GPP}) \qquad i\eta(\hat{u},\hat{v}) + (1+i\eta)^{\alpha-1}(\nabla\hat{u},\nabla\hat{v}) = (\hat{f},\hat{v}) \qquad \forall\hat{v}\in\tilde{H}_0^1,$$

$$(\widehat{GPPC})$$
$$i\eta(\hat{U}^h,\hat{v}^h) + (1+i\eta)^{\alpha-1}(\nabla\hat{U}^h,\nabla\hat{v}^h) = (\hat{f},\hat{v}^h)_N \qquad \forall\hat{v}^h\in\tilde{P}_N^0,$$

$$(\widehat{VK})$$
$$-\eta^2(\hat{u},\hat{v}) + (\nabla\hat{u},\nabla\hat{v}) + i\eta(1+i\eta)^{\alpha-1}(\nabla\hat{u},\nabla\hat{v}) = (\hat{f},\hat{v}), \qquad \forall\hat{v}\in\tilde{H}_0^1,$$

$$(\widehat{VKC})$$
$$-\eta^2(\hat{U}^h,\hat{v}^h) + (\nabla\hat{U}^h,\nabla\hat{v}^h) + i\eta(1+i\eta)^{\alpha-1}(\nabla\hat{U}^h,\nabla\hat{v}^h) = (\hat{f},\hat{v}^h)_N, \qquad \forall\hat{v}^h\in\tilde{P}_N^0.$$

Let $\hat{u}$ and $\hat{U}^h$ be the solution of $(\widehat{GPP})$ and $(\widehat{GPPC})$ or $(\widehat{VK})$ and $(\widehat{VKC})$, we set

$$\hat{\epsilon}^h = \hat{U}^h - \hat{w}^h \qquad : \qquad \hat{e}^h = \hat{u} - \hat{w}^h.$$

Now we put $\hat{v}=\hat{v}^h$ in $(\widehat{GPP})$ and $(\widehat{VK})$, and subtract $(\widehat{GPP})$ from $(\widehat{GPPC})$ or $(\widehat{VK})$ from $(\widehat{VKC})$. we have

$$\text{(I)} \quad i\eta(\hat{\epsilon}^h,\hat{v}^h) + (1+i\eta)^{\alpha-1}(\nabla\hat{\epsilon}^h,\nabla\hat{v}^h) = i\eta(\hat{e}^h,\hat{v}^h) + (1+i\eta)^{\alpha-1}(\nabla\hat{e}^h,\nabla\hat{v}^h) + (\hat{f},\hat{v}^h)_N - (\hat{f},\hat{v}^h).$$

$$\text{(II)} \quad -\eta^2(\hat{\epsilon}^h,\hat{v}^h) + (\nabla\hat{\epsilon}^h,\nabla\hat{v}^h) + i\eta(1+i\eta)^{\alpha-1}(\nabla\hat{\epsilon}^h,\nabla\hat{v}^h) = -\eta^2(\hat{e}^h,\hat{v}^h) + (\nabla\hat{e}^h,\nabla\hat{v}^h) + i\eta(1+\eta^2)^{\alpha-}(\nabla\hat{e}^h,\nabla\hat{v}^h) + (\hat{f},\hat{v}^h)_N - (\hat{f},\hat{v}^h).$$

Proof of Theorem 2.6. We take $\hat{v}^h = \hat{\epsilon}^h \in \tilde{P}_N^0$in (I) and get

(2.2.1)

$$\begin{aligned} & i\eta \parallel \hat{\epsilon}^h \parallel_0^2 + (1+\eta^2)^{\frac{\alpha-1}{2}}(\lambda_\alpha + i\mu_\alpha) \parallel \nabla\hat{\epsilon}^h \parallel_0^2 \\ =& i\eta(\hat{\epsilon}^h, \hat{\epsilon}^h) + (1+\eta^2)^{\frac{\alpha-1}{2}}(\lambda_\alpha + i\mu_\alpha)(\nabla\hat{\epsilon}^h, \nabla\hat{\epsilon}^h) \\ & + (\hat{f}, \hat{\epsilon}^h)_N - (\tilde{I}_N\hat{f}, \hat{\epsilon}^h)_N + (\tilde{I}_N\hat{f}, \hat{\epsilon}^h) - (\hat{f}, \hat{\epsilon}^h) \\ =& i\eta(\hat{\epsilon}^h, \hat{\epsilon}^h) + (1+\eta^2)^{\frac{\alpha-1}{2}}(\lambda_\alpha + i\mu_\alpha)(\nabla\hat{\epsilon}^h, \nabla\hat{\epsilon}^h) \\ & + (\hat{f} - \tilde{I}_N\hat{f}, \epsilon^h)_N + (\tilde{I}_N\hat{f} - \hat{f}, \hat{\epsilon}^h). \end{aligned}$$

(i) For bounded set $| \eta | \leq K$:
The real part of left hand side is bounded below as follows,

$$(1+\eta^2)^{\frac{\alpha-1}{2}} \parallel \nabla\hat{\epsilon}^h \parallel_0^2 \geq \delta_\alpha(K) \parallel \hat{\epsilon}^h \parallel_1^2 . \tag{2.2.2}$$

(see [4] and [11] for details)

On the other hand we can bound the right hand side above by

$$C(\parallel \hat{\epsilon}^h \parallel_0 \parallel \hat{\epsilon}^h \parallel_0 + \parallel \hat{\epsilon}^h \parallel_1 \parallel \epsilon^h \parallel_1 + \parallel \hat{f} - \tilde{I}_N\hat{f} \parallel_0 \parallel \hat{\epsilon}^h \parallel_0)$$

Thus

$$\begin{aligned} \parallel \hat{\epsilon}^h \parallel_1^2 &\leq C(\parallel \hat{\epsilon}^h \parallel_0 \parallel \hat{\epsilon}^h \parallel_0 + \parallel \hat{\epsilon}^h \parallel_1 \parallel \hat{\epsilon}^h \parallel_1 + \parallel \hat{f} - \tilde{I}_N\hat{f} \parallel_0 \parallel \hat{\epsilon}^h \parallel_0) \\ &\leq C(\parallel \hat{\epsilon}^h \parallel_1 \parallel \hat{\epsilon}^h \parallel_1 + \parallel \hat{f} - \tilde{I}_N\hat{f} \parallel_0 \parallel \hat{\epsilon}^h \parallel_1). \end{aligned}$$

$$\parallel \hat{\epsilon}^h \parallel_0 \leq \parallel \hat{\epsilon}^h \parallel_1 \leq C(\parallel \hat{\epsilon}^h \parallel_1 + \parallel \hat{f} - \tilde{I}_N\hat{f} \parallel_0), \qquad | \eta | \leq K. \tag{2.2.3}$$

(ii) For large $| \eta | \geq K$:
Taking the real part in (2.2.1), since $(1+\eta^2)^{\frac{\alpha-1}{2}} \approx | \eta |^{\alpha-1}$ we have

(2.2.4)

$$\begin{aligned} | \eta |^{\alpha-1} \parallel \hat{\epsilon}^h \parallel_1^2 \leq \frac{1}{\lambda_\alpha}(| \eta | \parallel \hat{\epsilon}^h \parallel_0 \parallel \hat{\epsilon}^h \parallel_0 + | \eta |^{\alpha-1} \parallel \hat{\epsilon}^h \parallel_1 \parallel \hat{\epsilon}^h \parallel_1 \\ + C \parallel \hat{f} - \tilde{I}_N\hat{f} \parallel_0 \parallel \hat{\epsilon}^h \parallel_0). \end{aligned}$$

Next we take the imaginary part and obtain the estimate

(2.2.5)

$$| \eta ||| \hat{\epsilon}^h \|_0^2 \le C(| \eta |^{\alpha-1} \| \hat{\epsilon}^h \|_1^2 + | \eta ||| \hat{e}^h \|_0 \| \hat{\epsilon}^h \|_0 + | \eta |^{\alpha-1} \| \hat{e}^h \|_1 \| \hat{\epsilon}^h \|_1 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0).$$

Using algebraic inequality,

$$ab \le \rho a^2 + \frac{1}{4\rho} b^2, \qquad \text{for any } a, b \in R, 0 < \rho \le 1,$$

and from (2.2.4) and (2.2.5)

(2.2.6)

$$\begin{aligned} &| \eta ||| \hat{\epsilon}^h \|_0^2 + | \eta |^{\alpha-1} \| \hat{\epsilon}^h \|_1^2 \\ \le &\frac{C}{\lambda_\alpha}(| \eta ||| \hat{e}^h \|_0 \| \epsilon^h \|_0 + | \eta |^{\alpha-1} \| \hat{e}^h \|_1 \| \epsilon^h \|_1 \\ &+ \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0) \\ \le &\frac{C}{\lambda_\alpha}(| \eta ||| \hat{e}^h \|_0^2 + | \eta |^{\alpha-1} \| \hat{e}^h \|_1^2 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0^2). \end{aligned}$$

Combining (2.2.3) and (2.2.6) gives

(2.2.7)

$$\begin{aligned} &(1+\eta^2)^{\frac{1}{2}} \| \hat{\epsilon}^h \|_0^2 + (1+\eta^2)^{\frac{\alpha-1}{2}} \| \hat{\epsilon}^h \|_1^2 \\ \le &\frac{C}{\lambda_\alpha}((1+\eta^2)^{\frac{1}{2}} \| \hat{e}^h \|_0^2 + (1+\eta^2)^{\frac{\alpha-1}{2}} \| \hat{e}^h \|_1^2 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0^2). \end{aligned}$$

It is that

$$\| \hat{\epsilon}^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} \le \frac{C}{\lambda_\alpha}(\| \hat{e}^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} + \| \hat{f} - \tilde{I}_N \hat{f} \|_0). \tag{2.2.8}$$

By the triangle inequality and (2.2.8),

$$\| \hat{u} - \hat{U}^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} \le \frac{C}{\lambda_\alpha}(\| \hat{e}^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} + \| \hat{f} - \tilde{I}_N \hat{f} \|_0).$$

Finally integration of both sides yields

$$\| u - U^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} \leq C_\alpha(\inf_{w^h \in \mathcal{H}^h_{\frac{1}{2},\frac{\alpha-1}{2}}} \| u - w^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} + \| f - I_N f \|_0)$$
$$\leq C_\alpha(\inf_{w^h \in \mathcal{H}^h_{\frac{1}{2},\frac{\alpha-1}{2}}} \| u - w^h \|_{\frac{1}{2},\frac{\alpha-1}{2}} + h^{\sigma-\frac{1}{2}} \| f \|_\sigma).///$$

Proof of Theorem 2.7. We take $\hat{v}^h = \hat{\epsilon}^h \in \hat{P}^0_N$ in (II) and get

(2.2.9)

$$\begin{aligned} &- \eta^2 \| \hat{\epsilon}^h \|_0^2 + (\nabla \hat{\epsilon}^h, \nabla \hat{\epsilon}^h) + i\eta(1+i\eta)^{\alpha-1}(\nabla \hat{\epsilon}^h, \nabla \hat{\epsilon}^h) \\ = &- \eta^2(\hat{e}^h, \hat{\epsilon}^h) + (\nabla \hat{e}^h, \nabla \hat{\epsilon}^h) + i\eta(1+i\eta)^{\alpha-1}(\nabla \hat{e}^h, \nabla \hat{\epsilon}^h) \\ &+ (\hat{f} - \tilde{I}_N \hat{f}, \hat{\epsilon}^h)_N + (\tilde{I}_N \hat{f} - \hat{f}, \hat{\epsilon}^h). \end{aligned}$$

(i) For large $| \eta | \geq K$:
Note that

$$| \eta | (1+\eta^2)^{\frac{\alpha-1}{2}} [-\cos(\alpha-1)\theta + i \sin(\alpha-1)\theta] \approx | \eta |^\alpha (-\mu_\alpha + i\lambda_\alpha).$$

We first take the imaginary part and obtain

(2.2.10)

$$\begin{aligned} | \eta |^\alpha \| \hat{\epsilon}^h \|_1^2 \leq \frac{C}{\lambda_\alpha}(&\eta^2 \| \hat{e}^h \|_0 \| \hat{\epsilon}^h \|_0 + (1 + | \eta |^\alpha) \| \hat{e}^h \|_1 \| \hat{\epsilon}^h \|_1 \\ &+ \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0). \end{aligned}$$

Taking the real part yields

(2.2.11)

$$\begin{aligned} \eta^2 \| \hat{e}^h \|_0^2 \leq & C((1 + | \eta |^\alpha) \| \hat{\epsilon}^h \|_1^2 + \eta^2 \| \hat{e}^h \|_0 \| \hat{\epsilon}^h \|_0 \\ &+ (1 + | \eta |^\alpha) \| \hat{e}^h \|_1 \| \hat{\epsilon}^h \|_1 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0) \\ \leq & C(\eta^2 \| \hat{e}^h \|_0 \| \hat{\epsilon}^h \|_0 + (1 + | \eta |^\alpha) \| \hat{e}^h \|_1 \| \hat{\epsilon}^h \|_1 \\ &+ \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0). \end{aligned}$$

Add (2.2.10) and (2.2.11) to get

(2.2.12)

$$\eta^2 \| \hat{\epsilon}^h \|_0^2 + | \eta |^\alpha \| \hat{\epsilon}^h \|_1^2 \le C(\eta^2 \| \hat{e}^h \|_0 \| \hat{\epsilon}^h \|_0 + (1+ | \eta |^\alpha) \| \hat{e}^h \|_1 \| \hat{\epsilon}^h \|_1 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0 \| \hat{\epsilon}^h \|_0)$$
$$\le C(\eta^2 \| \hat{e}^h \|_0^2 + (1+ | \eta |^\alpha) \| \hat{e}^h \|_1^2 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0^2).$$

(ii) For small $| \eta | \le K$:

(2.2.13) $$\| \hat{\epsilon}^h \|_0 \le \| \hat{\epsilon}^h \|_1 \le C_\alpha(N)(\| \hat{e}^h \|_1 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0).$$

Combining (2.2.12) and (2.2.13) gives

$$(1+\eta^2) \| \hat{\epsilon}^h \|_0^2 + (1+\eta^2)^{\frac{\alpha}{2}} \| \hat{\epsilon}^h \|_1^2$$
$$\le C((1+\eta^2) \| \hat{e}^h \|_0^2 + (1+\eta^2)^{\frac{\alpha}{2}} \| \hat{e}^h \|_1^2 + \| \hat{f} - \tilde{I}_N \hat{f} \|_0^2),$$

from which

(2.2.14) $$\| \hat{\epsilon}^h \|_{1,\frac{\alpha}{2}} \le C_\alpha(\| \hat{e}^h \|_{1,\frac{\alpha}{2}} + \| \hat{f} - \tilde{I}_N \hat{f} \|_0).$$

The same calculation as before yields the conculusion

$$\| u - U^h \|_{1,\frac{\alpha}{2}} \le C_\alpha(\inf_{w^h \in \mathcal{H}^h_{1,\frac{\alpha}{2}}} \| u - w^h \|_{1,\frac{\alpha}{2}} + \| f - I_N f \|_0)$$
$$\le C_\alpha(\inf_{w^h \in \mathcal{H}^h_{1,\frac{\alpha}{2}}} \| u - w^h \|_{1,\frac{\alpha}{2}} + h^{\sigma - \frac{1}{2}} \| f \|_\sigma). \qquad ///$$

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