A CERTAIN SUBGROUP OF
THE FUNDAMENTAL GROUP
OF A TRANSFORMATION GROUP

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1. Introduction

F.Rhodes [2] introduced the fundamental group $\sigma(X, x_0, G)$ of a
transformation group $(X, G)$ as an extension of the fundamental group
of a topological space $X$. B.J.Jiang [1] introduced the Jiang subgroup
$J(f, x_0)$ of the fundamental group $\pi_1(X, f(x_0))$ of a topological space
$X$ and a self-map $f$ of $X$ and showed that $J(f, x_0)$ is contained in
$Z(f_*(\pi_1(X, x_0)), \pi_1(X, f(x_0)))$. M.H.Woo and S.H.Han [3] introduced
the extended Jiang subgroup $HJ(f, x_0, G)$ of the fundamental group of
a transformation group $(X, G)$. But it is unknown that $J(f, x_0, G)$ is
contained in $Z(f_*(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$.

In this paper, we want to find a subgroup $HJ(f, x_0, G)$ of the ex-
tended Jiang subgroup of a transformation group which is contained
in $Z(f_*(\sigma(X, x_0, G)), \sigma(X, f(x_0), G))$ and is an extension of the Jiang
subgroup $J(f, x_0)$. That is, if the acting group $G$ is the trivial group
$\{1_X\}$, then this is the Jiang's result.

2. Definitions and Results

Let $(X, G, \pi)$ be a transformation group, where $X$ is a path connected
space with $x_0$ as base point. Given an element $g$ of $G$, a path $\alpha$ of order $g$
with base point $x_0$ is a continuous map $\alpha : I \rightarrow X$ such that $\alpha(0) = x_0$
and $\alpha(1) = gx_0$. A path $\alpha_1$ of order $g_1$ and a path $\alpha_2$ of order $g_2$
give rise to a path $\alpha_1 + g_1\alpha_2$ of order $g_1g_2$ defined by the equations

$$(\alpha_1 + g_1\alpha_2)(s) = \begin{cases} \alpha_1(2s), & 0 \leq s \leq 1/2 \\ g_1\alpha_2(2s - 1), & 1/2 \leq s \leq 1. \end{cases}$$

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Two paths $\alpha$ and $\alpha'$ of the same order $g$ are said to be homotopic if there is a continuous map $F : I^2 \rightarrow X$ such that

$$F(s, 0) = \alpha(s) \quad 0 \leq s \leq 1,$$
$$F(s, 1) = \alpha'(s) \quad 0 \leq s \leq 1,$$
$$F(0, t) = x_0 \quad 0 \leq t \leq 1,$$
$$F(1, t) = gx_0 \quad 0 \leq t \leq 1.$$

The homotopy class of a path $\alpha$ of order $g$ was denoted by $[\alpha; g]$. Two homotopy classes of paths of different orders $g_1$ and $g_2$ are distinct, even if $g_1 x_0 = g_2 x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $\circ$ form a group, where $\circ$ is defined by $[\alpha_1; g_1] \circ [\alpha_2; g_2] = [\alpha_1 + g_1 \alpha_2; g_1 g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the fundamental group of $(X, G)$ with base point $x_0$.

Let $f$ be a selfmap of $X$. A homotopy $H : X \times I \rightarrow X$ is said to be an $f$-cyclic homotopy if $H(\cdot, 0) = f = H(\cdot, 1)$. In this case, the path $H(x_0, \cdot)$ is called a trace of the homotopy $H$. In [2], Jiang has defined $J(f, x_0) = \{ [h] \in \pi_1(X, f(x_0)) \mid h \text{ is homotopic to a trace of an } f\text{-cyclic homotopy} \}$. For a transformation group $(X, G)$, a homotopy $H : X \times I \rightarrow X$ is said to be an $f$-cyclic homotopy of order $g$ if $H(\cdot, 0) = f, H(\cdot, 1) = gf$. In [3], Woo and Han has defined $J(f, x_0, G) = \{ [\alpha; g] \in \sigma(X, f(x_0), G) \mid \alpha \text{ is homotopic to a trace of an } f\text{-cyclic homotopy of order } g \}$. In this paper, the acting group $G$ is abelian.

**Definition 1.** $H J(f, x_0, G) = \{ [\alpha; g] \in \sigma(X, f(x_0), G) \mid \alpha \text{ is homotopic to a trace of an } f\text{-cyclic homotopy } H \text{ of order } g \text{ such that } H(g' x_0, \cdot) = g' H(x_0, \cdot) \text{ for each } g' \in G \}$.

**Theorem 1.** Let $(X, G)$ be a transformation group and $f$ be an endomorphism of $(X, G)$. Then $H J(f, x_0, G)$ is a subgroup of $\sigma(X, f(x_0), G)$.

**Proof.** Let $[\alpha_1; g_1]$ and $[\alpha_2; g_2]$ be any elements of $H J(f, x_0, G)$. Then there exist $f$-cyclic homotopies $K, K'$ of order $g_1, g_2$ with trace $\alpha_1, \alpha_2$ respectively such that $K(g' x_0, \cdot) = g' K(x_0, \cdot)$ and $K'(g' x_0, \cdot) = g' K'(x_0, \cdot)$ for each $g' \in G$. Define a homotopy $F : X \times I \rightarrow X$ by

$$F(x, t) = \begin{cases} 
K(x, 2t), & 0 \leq t \leq 1/2 \\
g_1 K'(x, 2t - 1), & 1/2 \leq t \leq 1.
\end{cases}$$
This is well defined and continuous. Now we have \( F(x, 0) = f(x) \) and \( F(x, 1) = g_1K'(x, 1) = g_1 g_2 f(x) \). It is easy to show that the trace of \( F \) is \( \alpha_1 + g_1 \alpha_2 \) and \( F(g'x_0, \cdot) = g'F(x_0, \cdot) \) for each \( g' \in G \). This means that \([\alpha_1; g_1] \circ [\alpha_2; g_2] \) belongs to \( HJ(f, x_0, G) \). Next, we show that if \([\alpha; g] \in HJ(f, x_0, G)\), then \([\alpha; g]^{-1} \in HJ(f, x_0, G)\). Let \([\alpha; g] \in HJ(f, x_0, G)\). Then there exists an \( f \)-cyclic homotopy \( K \) of order \( g \) with trace \( \alpha \) such that \( K(g'x_0, \cdot) = g'K(x_0, \cdot) \). Define \( F(x, t) = g^{-1}K(x, 1 - t) \). Then \( F \) is an \( f \)-cyclic homotopy of order \( g^{-1} \) with trace \( g^{-1} \alpha \rho \). Thus \([\alpha; g]^{-1} \) belongs to \( HJ(f, x_0, G) \).

If we take the acting group \( G = \{1_X\} \), then we obtain \( HJ(f, x_0, G) = J(f, x_0) \). From this fact, \( HJ(f, x_0, G) \) is an extension of the concept of Jiang subgroup \( J(f, x_0) \). Let \( f \) be a homomorphism from \((X, G)\) to \((X', G)\). Then \( f \) induces a homomorphism \( f_\sigma : \sigma(X, x_0, G) \rightarrow \sigma(X', f(x_0), G) \) given by \( f_\sigma([\alpha; g]) = [f\alpha; g] \).

**Theorem 2.** If \( f : (X, G) \rightarrow (X, G) \) is an endomorphism, then \( HJ(f, x_0, G) \) is contained in \( Z(f_\sigma(\sigma(X, x_0, G)), \sigma(X, f(x_0), G)) \).

**Proof.** Let \([\alpha; g]\) be any element of \( HJ(f, x_0, G) \). Then there exists an \( f \)-cyclic homotopy \( K : X \times I \rightarrow X \) of order \( g \) with trace \( \alpha \) such that \( K(g'x_0, \cdot) = g'K(x_0, \cdot) \) for each \( g' \in G \). For each \([\beta; g'] \in \sigma(X, x_0, G)\), we must show that \([\alpha; g] \circ (f_\sigma([\beta; g']))) = (f_\sigma([\beta; g']))) \circ [\alpha; g] \). That is, \( \alpha + g f \beta \) is homotopic to \( f \beta + g' \alpha \). Let \( J : I \times I \rightarrow X \) be given by \( J = K(\beta \times 1) \). Define \( F : I \times I \rightarrow X \) by

\[
F(s, t) = \begin{cases} J(2s(1 - t), 2st), & 0 \leq s \leq 1/2 \\ J(1 - (2 - 2s)t, (2 - 2s)t + 2s - 1), & 1/2 \leq s \leq 1 \end{cases}
\]

Then

\[
F(s, 0) = \begin{cases} J(2s, 0), & 0 \leq s \leq 1/2 \\ J(1, 2s - 1), & 1/2 \leq s \leq 1 \end{cases}
\]

\[
= \begin{cases} K(\beta(2s), 0), & 0 \leq s \leq 1/2 \\ K(g'x_0, 2s - 1), & 1/2 \leq s \leq 1 \end{cases}
\]

\[
= (f \beta + g' \alpha)(s),
\]

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\[
F(s, 1) = \begin{cases} 
J(0, 2s), & 0 \leq s \leq 1/2 \\
J(2s - 1, 1), & 1/2 \leq s \leq 1
\end{cases}
\]

\[
= \begin{cases} 
K(x_0, 2s), & 0 \leq s \leq 1/2 \\
K(\beta(2s - 1), 1), & 1/2 \leq s \leq 1
\end{cases}
\]

\[= (\alpha + g \beta f)(s),\]

\[F(0, t) = J(0, 0) = K(x_0, 0) = f(x_0) \text{ and } F(1, t) = J(1, 1) = K(gx_0, 1) = g(f(g'x_0)) = gg'f(x_0). \text{ Therefore we obtain } [\alpha + g \beta f; gg'] = [f \beta + g' \alpha; gg'].\]

Let \( \alpha : I \longrightarrow X \) be a path such that \( \alpha(0) = x_0 \) and \( \alpha(1) = x_1 \). Then \( \alpha \) induces an isomorphism \( \alpha_* : \sigma(X, x_0, G) \longrightarrow \sigma(X, x_1, G) \) such that \( \alpha_*([\beta; g]) = [\alpha \rho + \beta + g \alpha; g]. [2] \)

**Theorem 3.** Let \( F : X \times I \longrightarrow X \) be a homotopy from \( f \) to \( f' \) such that \( F(g'x_0, \cdot) = g'F(x_0, \cdot) \), then \( HJ(f, x_0, G) \) and \( HJ(f', x_0, G) \) are isomorphic.

**Proof.** Let \( F : X \times I \rightarrow X \) be a homotopy between \( f \) and \( f' \) such that \( F(g'x_0, \cdot) = g'F(x_0, \cdot) \). Let \( p(t) = F(x_0, t) \) for all \( t \in I \). Then \( p \) is a path from \( f(x_0) \) to \( f'(x_0) \). Since \( p_* : \sigma(X, f(x_0), G) \rightarrow \sigma(X, f'(x_0), G) \) is an isomorphism, it is sufficient to show that

\[p_*(HJ(f, x_0, G)) \subset HJ(f', x_0, G).\]

Let \( [\alpha; g] \) be any element of \( HJ(f, x_0, G) \). Then there exists an \( f \)-cyclic homotopy \( G : X \times I \longrightarrow X \) of order \( g \) with trace \( \alpha \) such that \( G(\cdot, 0) = f, G(\cdot, 1) = gf \) and \( G(g'x_0, \cdot) = g'G(x_0, \cdot) \) for each \( g' \in G \). Consider a homotopy \( K : X \times I \longrightarrow X \) given by

\[
K(x, t) = \begin{cases} 
F(x, 1 - 3t), & 0 \leq t \leq 1/3 \\
G(x, 3t - 1), & 1/3 \leq t \leq 2/3 \\
gF(x, 3t - 2), & 2/3 \leq t \leq 1.
\end{cases}
\]

Then \( K(x, 0) = F(x, 1) = f'(x), K(x, 1) = gF(x, 1) = gf'(x) \),

\[
K(x_0, t) = \begin{cases} 
F(x_0, 1 - 3t), & 0 \leq t \leq 1/3 \\
G(x_0, 3t - 1), & 1/3 \leq t \leq 2/3 \\
gF(x_0, 3t - 2), & 2/3 \leq t \leq 1.
\end{cases}
\]

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\[ = (p \rho + \alpha + gp)(t) = p_\ast(\alpha)(t), \]

and \( K(g'x_0, \cdot) = g'K(x_0, \cdot) \) for each \( g' \in G \). Thus \( p_\ast([\alpha; g]) \) belongs to \( HJ(f', x_0, G) \).

**Corollary 4.** If two selfmaps \( f, f' \) of \( X \) are homotopic, then \( J(f, x_0) \) is isomorphic to \( J(f', x_0) \).

**Theorem 5.** If \( f : (X, G) \longrightarrow (X, G) \) is an endomorphism and \( x_1 \) belongs to the orbit of \( x_0 \), then \( HJ(f, x_0, G) \) is isomorphic to \( HJ(f, x_1, G) \).

*Proof.* Let \( p \) be a path in \( X \) from \( x_0 \) to \( x_1 \). Then \( fp \) is a path from \( f(x_0) \) to \( f(x_1) \). Since \( (fp)_\ast : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, f(x_1), G) \) is an isomorphism, it is sufficient to show \( (fp)_\ast(HJ(f, x_0, G)) \subset HJ(f, x_1, G) \). Let \([\alpha; g]\) be any element of \( HJ(f, x_0, G) \). Then there is an \( f \)-cyclic homotopy \( K : X \times I \longrightarrow X \) of order \( g \) with trace \( \alpha \) such that \( K(g'x_0, \cdot) = g'K(x_0, \cdot) \). Since \( (fp)_\ast([\alpha; g]) = [fp\rho + \alpha + gf; g] = [\beta; g] \) and \( K(g'x_1, \cdot) = K(hx_0, \cdot) = g'hK(\cdot, \cdot) = g'K(x_1, \cdot) \), where \( \beta = K(x_1, \cdot), x_1 = hx_0 \) for some \( h \in G \), therefore \( (fp)_\ast([\alpha; g]) \) belongs to \( HJ(f, x_1, G) \).

**Definition 2.** A transformation group \((X, G)\) is called a transformation \( H \)-group with base point \( x_0 \) if there exists a map \( \mu : X \times X \longrightarrow X \) such that \( \mu(gx, y) = \mu(x, gy) = g\mu(x, y) \) and \( \mu(x, x_0) = x = \mu(x_0, x) \).

**Theorem 6.** If \((X, G)\) is a transformation \( H \)-group with base point \( f(x_0) \), then we have \( HJ(f, x_0, G) = \sigma(X, f(x_0), G) \).

*Proof.* Let \([\alpha; g]\) be any element of \( \sigma(X, f(x_0), G) \) and \( \mu \) be the transformation \( H \)-group structure. Define \( K : X \times I \longrightarrow X \) by \( K = \mu(f \times \alpha) \). Then \( K(x, 0) = \mu(f(x), \alpha(0)) = \mu(f(x), f(x_0)) = f(x) \), \( K(x, 1) = \mu(f(x), gf(x_0)) = gf(x) \), \( K(x_0, t) = \mu(f(x_0), \alpha(t)) = \alpha(t) \) and \( K(g'x_0, \cdot) = g'K(x_0, \cdot) \) for each \( g' \in G \). Thus \([\alpha; g]\) belongs to \( HJ(f, x_0, G) \).

**Example.** Let \( X = G = S^1 \) and \( \pi : X \times G \longrightarrow X \) be given by \( \pi(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1 + \theta_2)} \). Then \((X, G, \pi)\) is a transformation group. Moreover, we can take a transformation \( H \)-group structure \( \mu : X \times X \longrightarrow X \) given by \( \mu(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1 + \theta_2)} \) with the base point 1. Thus we have from Theorem 6 that \( HJ(f, 1, S^1) = \sigma(S^1, f(1), S^1) \).

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Let \((X, G, \pi)\) be a transformation group and \(X^X\) be the function space with the compact open topology. Define \(H(X^X) = \{ f \in X^X \mid \text{for any } g \in G, fg = gf \text{ on the base point } x_0 \}\) with the subspace topology of \(X^X\). Then \((H(X^X), G, \pi')\) is a transformation group where \(\pi'(f, g) = gf\). Let \(p : X^X \rightarrow X\) be the evaluation map given by \(p(f) = f(x_0)\). Then \(p : (H(X^X), G) \rightarrow (X, G)\) is a homomorphism and induces a homomorphism \(p_\sigma : \sigma(H(X^X), f, G) \rightarrow \sigma(X, f(x_0), G)\) given by \(p_\sigma([\alpha; g]) = [p\alpha; g]\).

**Theorem 7.** Let \(X\) be a \(CW\)-complex. Then \(p_\sigma(\sigma(H(X^X), f, G)) = HJ(f, x_0, G)\).

**Proof.** Let \([\alpha; g]\) be any element of \(\sigma(H(X^X), f, G)\). Then \(\alpha : I \rightarrow H(X^X)\) is a path of order \(g\) with base point \(f\). Since \(\alpha : I \rightarrow H(X^X) \subset X^X, \alpha \equiv i \circ \alpha : I \rightarrow X^X\) is a continuous map and hence \(\alpha\) is a path from \(f\) to \(gf\), where \(i : H(X^X) \rightarrow X^X\) is the inclusion map. Then \(\phi(\alpha) : X \times I \rightarrow X\) given by \(\phi(\alpha)(x, t) = \alpha(t)(x)\) satisfies that \(\phi(\alpha)(x, 0) = \alpha(0)(x) = f(x), \phi(\alpha)(x, 1) = \alpha(1)(x) = gf(x), \phi(\alpha)(x, t) = \alpha(t)(x_0) = p\alpha(t)\) and \(\phi(\alpha)(g'x_0, t) = \alpha(t)(g'x_0) = g'\alpha(t)(x_0) = g'\phi(\alpha)(x_0, t)\). Thus \(\phi(\alpha) : X \times I \rightarrow X\) is an \(f\)-cyclic homotopy of order \(g\) with trace \(p\alpha\) such that \(\phi(\alpha)(g'x_0, t) = g'\phi(\alpha)(x_0, t)\) for each \(g' \in G\). Therefore, \(p_\sigma[\alpha; g] = [p\alpha; g]\) belongs to \(HJ(f, x_0, G)\).

Conversely, let \([\alpha; g]\) be any element of \(HJ(f, x_0, G)\). Then there exists an \(f\)-cyclic homotopy \(F : X \times I \rightarrow X\) of order \(g\) with trace \(\alpha\) and \(F(g'x_0, t) = g'F(x_0, t)\). Since we can define \(\tilde{F} : I \rightarrow X^X\) by \(F(t) = F(\cdot, t), F(t)(g'x_0) = F(g'x_0, t) = g'F(x_0, t) = g'F(t)(x_0)\) for any \(g' \in G\). Thus \(\tilde{F}(t)\) belongs to \(H(X^X)\) and hence \([\tilde{F}; g]\) belongs to \(\sigma(H(X^X), f, G)\). Since \(p\tilde{F}(s) = F(s)(x_0) = F(x_0, s) = \alpha(s)\). Therefore \([\alpha; g]\) belongs to \(p_\sigma(\sigma(H(X^X), f, G))\).

**Theorem 8.** Let \(f, k\) be endomorphisms of \((X, G)\).

1. \(HJ(k, f(x_0), G) \subset HJ(kf, x_0, G)\),
2. \(k_\sigma(HJ(f, x_0, G)) \subset HJ(kf, x_0, G)\), where \(k_\sigma[\alpha; g] = [k\alpha; g]\).

**Proof.** (1) Let \([\alpha; g]\) be an element of \(HJ(k, f(x_0), G)\). Then there exists a \(k\)-cyclic homotopy \(K : X \times I \rightarrow X\) of order \(g\) with trace \(\alpha\) such that \(K(g'f(x_0), \cdot) = g'K(f(x_0), \cdot)\). If we define a homotopy \(K' : X \times I \rightarrow X\) by \(K' = K(f \times 1)\), then \(K'\) is a \(kj\)-cyclic homotopy.
of order $g$ with trace $\alpha$ such that $K'(g'x_0, \cdot) = g'K(x_0, \cdot)$. Thus $[\alpha; g]$ belongs to $HJ(kf, x_0, G)$.

(2) Let $[\alpha; g]$ be an element of $HJ(f, x_0, G)$ and $k_{\sigma}: \sigma(X, f(x_0), G) \rightarrow \sigma(X, kf(x_0), G)$ be a homomorphism. Then there exists an $f$-cyclic homotopy $K: X \times I \rightarrow X$ of order $g$ with trace $\alpha$ such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$. If we define $K': X \times I \rightarrow X$ by $K' = kK$, then $K'(x, 0) = kK(x, 0) = kf(x)$, $K'(x, 1) = kK(x, 1) = kf(x)$, $K'(x_0, t) = kK(x_0, t) = k \alpha(t)$ and $K'(g'x_0, \cdot) = g'K'(x_0, \cdot)$. Therefore, $k_{\sigma}([\alpha; g]) \in HJ(kf, x_0, G)$.

Remark. If we take $G = \{1_x\}$, (2) implies the Jiang’s result, that is, $k_{\pi}(J(f, x_0)) \subset J(kf, x_0)$.

Theorem 9. If $f, k: (X, G) \rightarrow (X, G)$ are isomorphisms and $f(x_0) = k(x_0)$, then $HJ(f, x_0, G)$ is equal to $HJ(k, x_0, G)$.

Proof. Let $[\alpha; g]$ be any element of $HJ(f, x_0, G)$. Then there exists an $f$-cyclic homotopy $K: X \times I \rightarrow X$ of order $g$ with trace $\alpha$ such that $K(g'x_0, \cdot) = g'K(x_0, \cdot)$. Define $K': X \times I \rightarrow X$ by $K' = K(f^{-1}k \times 1)$. Then $K'$ is a $k$-cyclic homotopy of order $g$ with trace $\alpha$ such that $K'(g'x_0, \cdot) = g'K'(x_0, \cdot)$. Thus $[\alpha; g]$ belongs to $HJ(k, x_0, G)$. Similarly, $HJ(k, x_0, G)$ is contained in $HJ(f, x_0, G)$. Therefore $HJ(f, x_0, G)$ is equal to $HJ(k, x_0, G)$.

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