SOME REMARKS ON PRIMAL IDEALS

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1. Introduction

Every ring considered in this paper will be assumed to be commutative and have a unit element. An ideal $A$ of a ring $R$ will be called primal if the elements of $R$ which are zero divisors modulo $A$, form an ideal of $R$, say $P$. If $A$ is a primal ideal of $R$, $P$ is called the adjoint ideal of $A$. The adjoint ideal of a primal ideal is prime [2]. The definition of primal ideals may also be formulated as follows: An ideal $A$ of a ring $R$ is primal if in the residue class ring $R/A$ the zero divisors form an ideal of $R/A$. If $Q$ is a primary ideal of a ring $R$ then every zero divisor of $R/Q$ is nilpotent; therefore, $Q$ is a primal ideal of $R$. That a primal ideal need not be primary, is shown by an example in [2].

Let $R[X]$ and $R[[X]]$ denote the polynomial ring and formal power series ring in an indeterminate $X$ over a ring $R$, respectively. Let $S$ be a multiplicative system in a ring $R$ and $S^{-1}R$ the quotient ring of $R$. Let $Q$ be a $P$-primary ideal of a ring $R$. Then $Q[X]$ is a $P[X]$-primary ideal of $R[X]$, and $S^{-1}Q$ is a $S^{-1}P$-primary ideal of a ring $S^{-1}R$ if $S\cap P = \emptyset$, and $Q[[X]]$ is a $P[[X]]$-primary ideal of $R[[X]]$ if $R$ is Noetherian [1]. We search for analogous results when primary ideals are replaced with primal ideals. To show an ideal $A$ of a ring $R$ to be primal, it suffices to show that $a - b$ is a zero divisor modulo $A$ whenever $a$ and $b$ are zero divisors modulo $A$.

**Definition.** An ideal $A$ of a $R$ is irreducible if $A$ can not be expressed as a finite intersection of proper divisors of $A$

A primal ideal may not be irreducible but every irreducible ideal is primal [2]. Without using this result, directly we can prove that if $A$ is an irreducible ideal of ring $R$, $A[X]$ is a primal ideal of $R[X]$. (First part of Proposition 1).


Proposition 1. Let \( A \) be an irreducible ideal of a ring \( R \). Then \( A[X] \) is a primal ideal of \( R[X] \). Furthermore, if \( P \) is the adjoint ideal of \( A \) considered as a primal ideal of \( R \), then \( P[X] \) is the adjoint ideal of \( A[X] \).

Proof. Let \( A \) be an irreducible ideal of \( R \). For each \( f(X) = \sum_{i=0}^{m} a_i X^i \in R[X] \), we define \( f(X) \) to be \( \sum_{i=0}^{m} a_i X^i \) where \( a_i = a_i + A \in R/A \) for each \( i = 0, \ldots, m \). Then \( f(X) \in R/A[X] \). Since the mapping \( \phi : R[X]/A[X] \to R/A[X] \) defined by \( \phi(f(X) + A[X]) = f(X) \) is an isomorphism and onto, we see that \( f(X) \) is a zero divisor modulo \( A[X] \) if and only if \( f(X) \) is a zero divisor in \( R/A[X] \). Let \( g(X) = \sum_{i=0}^{n} b_i X^i \) and \( h(X) = \sum_{i=0}^{p} c_i X^i \) be zero divisors modulo \( A[X] \). Then \( g(X) \) and \( h(X) \) are zero divisors in \( R/A[X] \).

By McCoy's theorem, there exist nonzero elements \( r = r + A \) and \( s = s + A \) in \( R/A \) such that \( \bar{r} \bar{g}(X) = 0 \) and \( \bar{s} \bar{h}(X) = 0 \). Clearly, \( (r) + A \) and \( (s) + A \) are proper divisors of \( A \); therefore, \( [(r) + A] \cap [(s) + A] \) is a proper divisor of \( A \). So there exists \( v \in [(r) + A] \cap [(s) + A] \) such that \( v \not\in A \). Then \( v = rt_1 + a_1 = st_2 + a_2 \) for some \( t_1, t_2 \in R \) and \( a_1, a_2 \in A \). Note that \( rt_1, st_2 \not\in A \) and \( v = rt_1 = st_2 \neq 0 \). But \( v(\bar{g}(X) - \bar{h}(X)) = \bar{r} \bar{t}_1 \bar{g}(X) - \bar{s} \bar{t}_2 \bar{h}(X) = 0 \); therefore, \( \bar{g}(X) - \bar{h}(X) \) is a zero divisor in \( R/A[X] \), so \( g(X) - h(X) \) is a zero divisor modulo \( A[X] \).

Thus \( A[X] \) is primal. Let \( P \) be an adjoint ideal of \( A \). We show that \( P[X] \) is the adjoint ideal of \( A[X] \).

Let \( f(X) = \sum_{i=0}^{n} a_i X^i \) be a zero divisor modulo \( A[X] \). Then \( f(X) = \sum_{i=0}^{n} \bar{a}_i X^i \) is a zero divisor in \( R/A[X] \) so there exists \( r \in R/A, r \neq 0 \) such that \( rf(X) = 0 \). Then all \( \bar{a}_i \) are zero divisors in \( R/A \) and all \( a_i \) are zero divisors modulo \( A \), so all \( a_i \) are in \( P \). So \( f(X) \in P[X] \), which implies that all zero divisors modulo \( A[X] \) are contained in \( P[X] \). Let \( q(X) = \sum_{i=0}^{n} d_i X^i \in P[X] \). We show that \( q(X) \) is a zero divisor modulo \( A[X] \). If \( q(X) \in A[X] \), then clearly \( q(X) \) is a zero divisor modulo \( A[X] \).

So we assume \( q(X) \not\in A[X] \). Suppose that \( d_1, d_2, \ldots, d_4 \not\in A \) and all other \( d_i \) are in \( A \). Then there exist \( t_1, t_2, \ldots, t_s \in P - A \) such that \( t_1 d_1, t_2 d_2, \ldots, t_s d_s \in A \). Let \( D = [(t_1) + A] \cap [(t_2) + A] \cap \cdots \cap [(t_s) + A] \), then \( D \) is a proper divisor of \( A \). Since \( A \) is irreducible, there exists \( d \in D \) such that \( d \not\in A \).

Then \( d = r_1 t_1 + a_1 = r_2 t_2 + a_2 = \cdots = r_s t_s + a_s \) for some \( r_1, r_2, \ldots, r_s \).
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$\in R$ and $a_1, a_2, \cdots, a_s \in A$. Since $t_1d_{i_1}, \cdots, t_sd_{i_s} \in A$, $d \cdot q(X) = d \cdot \sum_{i=0}^{n} d_i X^i \in A[X]$. Note that $d \not\in A[X]$. Thus $q(X)$ is a zero divisor modulo $A[X]$. We showed that every element of $P[X]$ is a zero divisor modulo $A[X]$. Thus $P[X]$ is the disjoint ideal of $A[X]$. 

2. Main Results

Naturally, the following question arises: If $A$ is a primal ideal of a ring $R$ with the adjoint prime ideal $P$, is $A[X]$ a primal ideal of $R[X]$ with the adjoint prime ideal of $P[X]$? In Theorem 1 we will see that the answer of this question is not affirmative.

A Noetherian ring has the property that annihilator of each ideal consisting entirely of zero divisors is nonzero [4; p.56]. Huckaba [3] abstracted this to arbitrary ring as following definition.

DEFINITION. A ring satisfies Property (*) if each finitely generated ideal consisting entirely of zero divisors has nonzero annihilator.

Every polynomial ring $R[X]$ satisfies Property (*) and every zero-dimensional ring satisfies Property (*)[3; p.7,9].

THEOREM 1. Let $A$ be a primal ideal of a ring $R$ with the adjoint prime ideal $P$. Then $R/A$ satisfies property (*), if and only if $A[X]$ is a prime ideal of $R[X]$ with the adjoint prime ideal $P[X]$.

Proof. Suppose that $R/A$ satisfies Property (*). Let $F(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{i=0}^{n} b_i X^i$ be zero divisors modulo $A[X]$. Then $f(x) = g(x)$ are zero divisors in $R/A[X]$, so $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n$ are zero divisors in $R/A$. Then $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n$ are zero divisors modulo $A$. Let $B = (a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n)$. Then $B \subseteq P$ since $P$ is an ideal and consists of all zero divisors modulo $A$. Then $B$ consists entirely of zero divisors modulo $A$, so the ideal $\bar{B} = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n)$ consists entirely of zero divisors of $R/A$. Since $R/A$ satisfies Property (*), there exists $\bar{r} \in R/A$, $\bar{r} \not= \bar{o}$ such that $\bar{r} \cdot \bar{B} = (\bar{o})$. Then $\bar{r}(\bar{f}(X) - \bar{g}(X)) = \bar{o}$, so $\bar{f}(X) - \bar{g}(X)$ is a zero divisor in $R/A[X]$ therefore, $f(X) - g(X)$ is a zero divisor modulo $A[X]$. Thus $A[X]$ is a prime ideal of $R[X]$.

Next, we show that $P[X]$ is the adjoint ideal of $A[X]$. That all zero divisors modulo $A[X]$ are contained in $P[X]$, can be proved in the
same way as in the proof of Proposition 1, so we omit its proof. Let \( q(x) = \sum_{i=0}^{n} d_i X^i \in P[X] \). We will show that \( q(X) \) is a zero divisor modulo \( A[X] \). Let \( D = (d_1, \cdots, d_n) \), then \( D \subseteq P \) and \( D \) consists entirely of zero divisors modulo \( A \). Then the ideal \( D = (\bar{d}_1, \cdots, \bar{d}_n) \) consists entirely of zero divisors in \( R/A \) where \( \bar{d}_i = d_i + A \in R/A \) for each \( i \). Since \( R/A \) satisfies Property \((\ast)\), there exists \( \bar{r} \) in \( R/A \), \( \bar{r} \neq 0 \) such that \( \bar{r} D = (0) \). Hence \( \bar{q}(x) = \sum_{i=0}^{n} \bar{d}_i X^i \) is a zero divisor in \( R/A[X] \); Therefore, \( q(x) \) is a zero divisor modulo \( A[X] \). Thus \( P[X] \) is an adjoint ideal of \( A[X] \).

Conversely, suppose that \( A[X] \) is a primal ideal of \( R[X] \) with its adjoint ideal \( P[X] \). Let \( \bar{U} = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n) \) be an ideals of \( R/A \) consisting entirely of zero divisors of \( R/A \). Then \( u_1, u_2, \cdots, u_n \) are zero divisors modulo \( A \). So \( \sum_{i=0}^{n} u_i X^i \in P[X] \). Then \( \sum_{i=0}^{n} u_i X^i \) is a zero divisor modulo \( A[X] \). Hence \( \sum_{i=0}^{n} \bar{u}_i X^i \) is a zero divisor of \( R/A[X] \). Then there exists \( \bar{\nu} \in R/A, \bar{\nu} \neq 0 \) such that \( \bar{\nu} u_i = 0 \) for each \( i = 0, 1, \cdots, n \). So \( \bar{\nu} \cdot \bar{U} = 0 \). Thus the ring \( R/A \) satisfies Property \((\ast)\).

**Corollary 1.** If \( A \) is an irreducible ideal of a ring \( R \), then \( R/A \) satisfies Property \((\ast)\).

**Proof.** Let \( A \) be an irreducible ideal of a ring \( R \) and \( P \) its adjoint ideal. By Proposition 1, \( A[X] \) is a primal ideal of \( R[X] \) with the adjoint ideal \( P[X] \). Then by Theorem 1, \( R/A \) satisfies Property \((\ast)\).

**Corollary 2.** Let \( A \) be a primal ideal of a ring \( R \) with the adjoint ideal \( P \). Then if \( R/A \) satisfies Property \((\ast)\), \( A[X_1, \cdots, X_n] \) is primal ideal of \( R[X_1, \cdots, X_n] \) with the adjoint ideal \( P[X_1, \cdots, X_n] \).

**Proof.** Let \( A \) be a primal ideal of a ring \( R \) with the adjoint ideal \( P \). Assume that \( R/A \) satisfies Property \((\ast)\). Then \( A[X_1] \) is a primal ideal of \( R[X_1] \) with the adjoint ideal \( P[X_1] \) (by Theorem 1). Since the polynomial ring \( R/A[X_1] \) satisfies Property \((\ast)\)[3,p.17] and \( R/A[X_1] \simeq R[X_1]/A[X_1] \), it follows that \( R[X_1]/A[X_1] \) satisfies \((\ast)\). Then by Theorem 1, \( A[X_1, X_2] \) is a primal ideal of \( R[X_1, X_2] \) with the adjoint ideal \( P[X_1, X_2] \).

**Theorem 2.** Let \( A \) be an ideal of a ring \( R \). Then if \( A[X] \) (resp. \( A[[X]] \)) is a primal ideal of \( R[X] \) (resp. \( R[[X]] \)), then \( A \) is primal.
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Proof. Let $a_1$ and $a_2$ be elements of $R$ which are zero divisors modulo $A$. Then there exist $b_1$ and $b_2$ in $R$ such that $b_1, b_2 \notin A, a_1b_1 \in A$ and $a_2b_2 \in A$. Then $b_1, b_2 \notin A[X], a_1b_1 \in A[X]$ and $a_2b_2 \in A[X]$, so $a_1$ and $a_2$ are zero divisors modulo $A[X]$. Since $A[X]$ is a primal ideal of $R[X]$, there exists $g(X) = \sum_{i=0}^{n} c_iX^i$ in $R[X]$ such that $g(X) \notin A[X]$ and $g(X)(a_1 - a_2) \in A[X]$. Then $c_j(a_1 - a_2) \in A$ for some $c_j \notin A$; therefore, $a_1 - a_2$ is a zero divisor modulo $A$, and $A$ is a primal ideal of $R$. Similarly, we can prove the Theorem when $A[X]$ and $R[X]$ are replaced by $A[[X]]$ and $R[[X]]$, respectively.

Theorem 3. Let $A$ be a primal ideal of $R$ with the adjoint ideal $P$ and let $S$ be a multiplicative system in $R$ such that $S \cap P$ is empty. Then $S^{-1}A$ is a primal ideal of $S^{-1}R$ with the adjoint ideal $S^{-1}P$.

Proof. We show that if $a/t$ is a zero divisor modulo $S^{-1}A$, then $a$ is a divisor modulo $A$. Let $a/t$ be a zero divisor modulo $S^{-1}A$, then there exist $b/s \in S^{-1}R$ such that $b/s \notin S^{-1}A$ and $(a/t) \cdot (b/s) \in S^{-1}A$. Then there exists $v$ in $S$ such that $abv \in A$. Clearly, $bv \notin A$, for otherwise $b/s \in S^{-1}A$ which violates $b/s \notin S^{-1}A$. Hence $a$ is a zero divisor modulo $A$. To prove $S^{-1}A$ to be primal, let $c_1/t_1$ and $a_2/t_2$ be zero divisors modulo $S^{-1}A$. Then $a_1$ and $a_2$ are zero divisors modulo $A$. Since $A$ is a primal ideal with adjoint ideal $P$, $a_1t_2 - a_2t_1 \in P$. Then there exists $r$ in $P - A$ such that $(a_1t_2 - a_2t)r \in A$.

Then $(a_1t_2 - a_2t_1)r/t_1t_2u = (a_1/t_1 - a_2/t_2)(r/u) \in S^{-1}A$ for any $u \in S$. Claim $r/u \notin S^{-1}A$. For suppose $r/u \in S^{-1}A$, then there exists $v$ in $S$ such that $vr \in A$. Since $r \in P - A$, $v$ is a zero divisor modulo $A$ so $v \in P$. Then $v \in S \cap P$ which violates our assumption $S \cap P = \emptyset$. Hence $r/u \notin S^{-1}A$ and $a_1/t_1 - a_2/t_2$ is a zero divisor modulo $S^{-1}A$. Therefore, $S^{-1}A$ is a primal ideal of $S^{-1}R$. Next we show that $S^{-1}P$ is the adjoint ideal of $S^{-1}A$.

Let $a/t$ be a zero divisor modulo $S^{-1}A$, then $a$ is a zero divisor modulo $A$; therefore, $a \in P$ and $a/t \in S^{-1}P$. This shows that every zero divisor modulo $S^{-1}A$ is contained in $S^{-1}P$. Let $b/s \in S^{-1}P$, then $bd \in P$ for some $d \in S$. Since $P$ is a prime ideal and $d \notin P$, it follows that $b \in P$ and $b$ is a zero divisor modulo $A$. Then there exists $c$ in $P - A$ such that $bc \in A$. Then $(b/s)(c/t) \in S^{-1}A$ for any $t \in S$. Claim $c/t \notin S^{-1}A$. For suppose $c/t \in S^{-1}A$, then $cv \in A$ for some $v \in S$. 75
Note that \( v \) is a regular element modulo \( A \), so \( c \in A \). But \( c \in P - A \) which leads a contradiction. So \( c/t \not\in S^{-1}A \) and \( b/s \) is a zero divisor modulo \( S^{-1}A \). This shows that every element of \( S^{-1}P \) is a zero divisor modulo \( S^{-1}A \). Thus we can conclude that \( S^{-1}P \) is the set of all zero divisors modulo \( S^{-1}A \); therefore, \( S^{-1}P \) is the adjoint ideal of \( S^{-1}A \).

Let \( A \) be an ideal of a ring \( R \) and \( S \) a multiplicative system in \( R \). Consider the mapping \( \phi: R \to S^{-1}R \) defined by \( \phi(a) = as/s \) for \( s \in S \). Then \( \phi \) is a ring homomorphism. Let \( S^{-1}A \cap R \) denote the complete inverse image of \( S^{-1}A \) under \( \phi \). Then \( S^{-1}A \cap R \) is the contraction of \( S^{-1}A \) to \( R \).

**Theorem 4.** Let \( A \) be an ideal of a ring \( R \) and \( S \) a multiplicative system in \( R \) such that \( S \cap A \) is empty. Then if \( S^{-1}A \) is a primal ideal of \( S^{-1}R \), then \( S^{-1}A \cap R \) is a primal ideal of \( R \).

**Proof.** Let \( A_S = \{ x \in R | sx \in A \text{ for some } s \in S \} \). Then it follows that \( S^{-1} \cap R = A_S \) [5; p.69]. Let \( a \) be a zero divisor modulo \( A_S \). Then there exists \( b \) in \( R \) such that \( b \not\in A_S \) and \( ab \in A \). Then \( sab \in A \) for some \( s \in S \). Then \( ab/s_1s_2 \in S^{-1}A \) for any \( s_1, s_2 \in S \). Claim \( b/s_2 \not\in S^{-1}A \). For suppose \( b/s_2 \in S^{-1}A \). Then \( tb \in A \) for some \( t \in S \), hence \( b \in A_S \) which violates \( b \not\in A_S \). So \( b/s_2 \not\in S^{-1}A \); therefore, \( a/s_1 \) is a zero divisor modulo \( S^{-1}A \) for any \( s \in S \). Let \( a_1 \) and \( a_2 \) be elements of \( R \) which are zero divisors modulo \( A_S \). Then \( t_1a_1 \) and \( t_2a_2 \) are zero divisors modulo \( A_S \) for any \( t_1, t_2 \in S \). Then \( t_1a_1/t_1 \) and \( t_2a_2/t_2 \) are zero divisors modulo \( S^{-1}A \). Since \( S^{-1}A \) is a primal ideal, there exists \( a_3/t_3 \) in \( S^{-1}R \) such that \( a_3/t_3 \not\in S^{-1}A \) and \( (a_1t_1/t_1 - a_2t_2/t_2)(a_3/t_3) \in S^{-1}A \). Then there exists \( t_4 \) in \( S \) such that \( (a_1t_1t_2 - a_2t_1t_2)a_3t_4 \in A \). Therefore, \( (a_1 - a_2)a_3 \in A_S \). Since \( a_3/t_3 \not\in S^{-1}A \), we see that \( a_3t \not\in A \) for any \( t \in S \). Then \( a_3 \not\in A_S \) so \( a_1 - a_2 \) is a zero divisor modulo \( A_S \). Thus \( S^{-1}A \cap R (= A_*) \) is a primal ideal of \( R \).

**References**


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