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HAAR MEASURES OF SOME SPECIFIC SETS ARISING FROM THE ELLIPTIC TORI

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1. Introduction

We let F be a p-adic field with ring of integers O. Suppose $\theta_i \in F^x \backslash (F^x)^2$ for $i = 1, 2$ and write $E^{\theta_i} := F(\sqrt{\theta_i})$. Then there appear some specific sets such as $(E^{\theta_i})^x / F^x$ in [1] which we need to measure.

In addition to that, another possible condition attached to the generalized results in [2] had better be presented even though they may not be quite so important.

This paper is concerned with these matters. Most notations and conventions are standard and have been used also in [1] and [2].

2. Measures of some specific sets

Let $E_1^{\theta_1} := \{x \in E^{\theta_i} : N_F^{E^{\theta_i}}(x) = 1\}$. We see $E_1^{\theta_i}/\{\pm 1\} \subset (E^{\theta_i})^x / F^x$ in §6 [1], which needs to be measured. We met with an elliptic torus T which is isomorphic to $E_1^{\theta_1} \times E_1^{\theta_2}$, i.e., T is the set of all F-points of the form $\begin{bmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\theta_1 & 0 & a & 0 \\ 0 & \beta\theta_2 & 0 & \alpha \end{bmatrix}$ with $a^2 - b^2\theta_1 = \alpha^2 - \beta^2\theta_2 = 1$.

The same is true for G-conjugates of T. Since the Haar measure of T is determined by that of $E_1^{\theta_i}$ and since we may have the relation $E_1^{\theta_i}/\{\pm 1\} \subset (E^{\theta_i})^x / F^x$, we see immediately that the measure of $E_1^{\theta_i}$ is determined by those of $(E^{\theta_i})^x$ and F^x.

Now we want to compute explicitly the measure of $(E^{\theta_i})^x / F^x$, which is finite. We use the following notations.

μ_i : Haar measure of $(E^{\theta_i})^x$ for $i = 1, 2$.
μ : Haar measure of F^x
π_i : prime element in $(E^{\theta_i})^x$

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e_i : ramification index
U_i : unit group in $(E^{\theta_i})^x$
U : unit group in F.
O_i : ring of integers in E^{θ_i}
P_i : maximal ideal in O_i
O : ring of integers in F
P : maximal ideal in O
q : cardinality of the set O/P which is not a power of 2
$\tilde{P}$: $= P^r$ for a fixed positive integer r.

Then we see easily that $F^x \simeq \{\pi_i^{e_i n}\} \times U$,

$$(E^{\theta_i})^x \simeq \{\pi_i^n\} \times U_i \quad \text{with} \quad n \in Z.$$

The Haar measure of the cyclic groups $\{\pi_i^n\}, \{\pi_i^{e_i n}\}$ are the counting measures.

Note here that $O_i \simeq O \times O$ and $[O_i/P_i : O/P] = \frac{2}{e_i}$. Since $O_i = U_i \dot{\cup} P_i$ and $O = U \dot{\cup} P$, we see $\mu(O) = \mu(U) + \mu(P) = 1$ and $\mu_i(O_i) = \mu(U_i) + \mu(P_i) = 1$ considering Haar measure normalizations. Here $\dot{\cup}$ indicates disjoint union. We see immediately that $\mu(U) = 1-q^{-1} \& \mu_i(U_i) = 1-q^{-\frac{2}{e_i}}$ because of $\mu(P) = q^{-1} \& \mu_i(P_i) = q^{-\frac{2}{e_i}}$. Next considering dimensions of the factors in the decomposition, we see that $(E^{\theta_i})^x/F^x \simeq \{\pi^n\}/\{\pi^{e_i n}\} \times U_i/U$, which gives rise to $\mu_i\{(E^{\theta_i})^x/F^x\} = e_i(1-q^{-\frac{2}{e_i}})/(1-q^{-1})$.

Incidentally we may also find the measure $\bar{\mu}$ of an elliptic torus $(E^{\theta_1})^x \times (E^{\theta_2})^x$ (in GL(4)) divided by the F-split torus contained in it; In other words, from the relation $(E^{\theta_1})^x \times (E^{\theta_2})^x/_{F^x \times F^x} \simeq (E^{\theta_1})^x/F^x \times (E^{\theta_2})^x/F^x$, we see immediately that

$$\begin{aligned}\bar{\mu}\{(E^{\theta_1})^x \times (E^{\theta_2})^x/_{F^x \times F^x}\} &= \mu_1\{(E^{\theta_1})^x/F^x\} \times \mu_2\{(E^{\theta_2})^x/F^x\} \\ &= e_1 e_2 (1-q^{-\frac{2}{e_1}})(1-q^{-\frac{2}{e_2}})/(1-q^{-1})^2.\end{aligned}$$

3. Generalized results and another possible condition

In the proof of proposition (6.1) in [1], we mentioned that whether or not **P** (resp. **Q**) belongs to $N_F^{E^{\theta_1}}[(E^{\theta_1})^x]$ (resp. $N_F^{E^{\theta_2}}[(E^{\theta_2})^x]$ does not depend upon $p \in \tilde{P}$. Here $\mathbf{P} = (1-p)^2 - 2a(1-p) + 1$, $\mathbf{Q} = (1-p)^2 - 2\alpha(1-p) + 1$ for any element p in $\tilde{P}$. But I found that **P** and **Q** actually belong to the norm groups. The reason is as follows: I claim that we may set $\sqrt{1 + b^2\theta_i} = 1 + \frac{1}{2}b^2(1+x)\theta_i$ for some $x \in \tilde{P}$. Squaring both sides and rearranging them, we have an equation with coefficients in O, $b^2\theta_i x^2 + (4 + 2b^2\theta_i)x + b^2\theta_i = 0$, which obviously has a solution $x \in \tilde{P}$ by Hensel's Lemma. Hence $\mathbf{P} = (1-p)^2 - 2a(1-p) + 1 = p^2 - (1-p)(1+x)b^2\theta_1 \in N_F^{E^{\theta_1}}((E^{\theta_1})^x)$. Similarly for **Q**.

Hence our conditions may reduce to the form:

$$\frac{\bar{a}b}{2(\alpha - a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^x]$$

and

$$\frac{\bar{a}\beta}{2(a - \alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^x].$$

I have recently generalized our results for any maximal torus $^\circ T$, which has been published in [2].

So reformulating our main results, we have

THEOREM. *Suppose that $^\circ T = T^h \cap G$ for some $h \in S_{p_4}(\bar{F})$, where T should be perceived to be contained in the same set $S_{p_4}(\bar{F})$.*

Then the regular germs $\Gamma_{u(\bar{a})}(^\circ t)$ with $^\circ t = t^h \in {}^\circ T$ associated to the representative unipotent matrices $u(\bar{a})$ are as follows:

Let

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 4(a-1)(\alpha-1) & 0 & 1 & 0 \\ -4(a-1)(\alpha-1) & 2(a+\alpha)-4 & -1 & 2(a+\alpha)-3 \end{bmatrix}$$

(i) If $X \in (t^h)^{Gd(\sqrt{\bar{a}})}$ with t^h sufficiently close to the identity, then $\Gamma_{u(\bar{a})}(^\circ t) = |D(^\circ t)|^{-\frac{1}{2}}$;

(ii) In particular in the case of $h \in G$, the above condition is reduced to the form: $\frac{\bar{a}b}{2(\alpha-a)} \in N_F^{E^{\theta_1}}[(E^{\theta_1})^x]$ and $\frac{\bar{a}\beta}{2(a-\alpha)} \in N_F^{E^{\theta_2}}[(E^{\theta_2})^x]$.

(iii) $\Gamma_{u(\bar{a})}({}^\circ t) = 0$, otherwise.

References

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