QUASI $O_2$-SPACES

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0. INTRODUCTION

It is known that the behavior of a certain family of subsets of a completely regular space characterizes its structure. We note among others that a completely regular space $X$ is an $O_2$-space (a basically disconnected space, a quasi F-space, resp.) iff every open set (cozero-set, dense cozero-set, resp.) of $X$ is $Z$-embedded ($C^*$-embedded, resp.) (see [1], [3], and [8]).

For $O_2$-spaces, it is also known that a completely regular space $X$ is an $O_2$-space iff $\nu X$ is an $O_2$-space and that the real line $\mathbb{R}$ is an $O_2$-space but $\beta \mathbb{R}$ is not an $O_2$-space ([2] and [7]).

In this paper, we introduce a concept of quasi $O_2$-spaces which generalizes that of $O_2$-spaces. Indeed, a completely regular space $X$ is a quasi $O_2$-space if for any regular closed set $A$ in $X$, there is a zero-set $Z$ in $X$ with $A = \text{cl}_X(\text{int}_X(Z))$. We then show that $X$ is a quasi $O_2$-space iff every open subset of $X$ is $Z^\#$-embedded and that $X$ is a quasi $O_2$-space iff $\beta X$ is a quasi $O_2$-space. Furthermore, it is shown that quasi $O_2$-spaces are left fitting with respect to covering maps.

Observing that a quasi $O_2$-space is an extremally disconnected iff it is a cloz-space, the minimal extremally disconnected cover, basically disconnected cover, quasi F-cover, and cloz-cover of a quasi $O_2$-space $X$ are all equivalent. Finally it is shown that a compactification $Y$ of a quasi $O_2$-space $X$ is again a quasi $O_2$-space iff $X$ is $Z^\#$-embedded in $Y$.

For the terminology, we refer to [6].

1. QUASI $O_2$-SPACES

In the following, we assume that every space is a completely regular space. For a space $X$, let $Z(X)$ ($R(X)$, resp.) denote the set of all zero-sets (regular closed sets, resp.) on $X$.

The following are introduced by Henriksen, Vermeer, and Woods [4].

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Notation 1.1. For any space $X$, let
(a) $Z(X)^# = \{ \, \text{cl}_X(\text{int}_X(A)) : A \in Z(X) \, \}.$
(b) $G(X) = \{ \, \text{cl}_X(C) : C \text{ is a cozero-set and there is a cozero-set } D \text{ in } X \text{ such that } C \cap D = \emptyset \text{ and } C \cup D \text{ is dense in } X \, \}$

Now we introduce the concept of quasi $O_2$-spaces.

Definition 1.2. A space $X$ is said to be a quasi $O_2$-space if $Z(X)^# = R(X)$.

Since a space $X$ is an $O_2$-space iff $R(X) \subseteq Z(X)$, and $Z(X)^# \subseteq R(X)$, every $O_2$-space is a quasi $O_2$-space. We recall that every perfectly normal space is an $O_2$-space and that an extremally disconnected space is an $O_2$-space (see [1]).

Proposition 1.3. A space $X$ is a quasi $O_2$-space iff $R(X) = Z(X)^# = G(X)$.

Proof. We first note that $G(X) = \{ \, A \in Z(X)^# : \text{cl}_X(X - A) \in Z(X)^# \, \}.$ (see [4]). Thus $G(X) \subseteq Z(X)^# \subseteq R(X)$. Suppose $R(X) = G(X)$, then $X$ is clearly a quasi $O_2$-space. For the converse, take any open set $U$ in $X$, then $\text{cl}_X(U)$, $\text{cl}_X(X - \text{cl}_X(U)) \in R(X) = Z(X)^#$; therefore $\text{cl}_X(U) \in G(X)$. Thus $R(X) = G(X)$.

Definition 1.4. Let $Y$ be a space, then a subspace $X$ of $Y$ is said to be $Z^\#$-embedded in $Y$ if for any $A \in Z(X)^#$, there is a $B \in Z(Y)^#$ with $A = B \cap X$. In case, the inclusion map $X \hookrightarrow Y$ is also said to be $Z^\#$-embedded.

We recall that a space $X$ is an $O_2$-space iff every open subset of $X$ is $Z$-embedded.

Theorem 1.5. For any space $X$, the following are equivalent:
(a) $X$ is a quasi $O_2$-space.
(b) Every open subset $U$ of $X$ is $Z^\#$-embedded in $X$.
(c) Every dense open subset $U$ of $X$ is $Z^\#$-embedded in $X$.

Proof. (a) $\implies$ (b) Take any $Z \in Z(U)$. Then there is a closed set $A$ in $X$ with $Z = A \cap U$. Since $U$ is open, $\text{cl}_U(\text{int}_U(Z)) = \text{cl}_X(\text{int}_X(A)) \cap U$ and $\text{cl}_X(\text{int}_X(A)) \in R(X) = Z(X)^#$. Thus $U$ is $Z^\#$-embedded in $X$.
(b) $\implies$ (c) It is trivial.
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(c) $\implies$ (a) Take any open set $U$ in $X$. Put $S = U \cup (X - \text{cl}_X(U))$. Then $S$ is open dense in $X$. Define a map $f : S \rightarrow R$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in X - \text{cl}_X(U)$. Then $f$ is continuous, $f^{-1}(0) = U$ and hence $U$ is a zero-set in $S$. Since $S$ is open dense in $X$, there is a zero-set $Z$ in $X$ such that $\text{cl}_S(\text{int}_S(U)) = \text{cl}_X(U) \cap S = \text{cl}_X(\text{int}_X(Z)) \cap S$. Since $S$ is dense in $X$ and $\text{cl}_X(U)$, $\text{cl}_X(\text{int}_X(Z))$ are regular closed sets in $X$, $\text{cl}_X(U) = \text{cl}_X(\text{int}_X(Z))$. Hence $R(X) \subseteq Z(X)^\#$, so that $X$ is a quasi $O_z$-space.

**Lemma 1.6.** Let $X$ be a space and $U \subseteq F \subseteq X$. If $U$ is an open $Z^\#$-embedded subset of $X$, then $U$ is $Z^\#$-embedded in $F$.

**Proof.** Take any $A \in Z(U)^\#$. Since $U$ is $Z^\#$-embedded in $X$, there is a $B \in Z(X)$ with $A = \text{cl}_X(\text{int}_X(B)) \cap U$. Since $B \cap F$ is closed in $F$ and $U$ is open in $F$, $\text{cl}_F(\text{int}_F(B \cap F)) \cap U = \text{cl}_U(\text{int}_U(B \cap U)) = \text{cl}_X(\text{int}_X(B)) \cap U = A$. Since $B \cap F \in Z(F)$, $U$ is $Z^\#$-embedded in $F$.

The first half of the following is immediate from Lemma 1.6 and the second half also follows from the routine calculation.

**Proposition 1.7.** Let $X$ be a quasi $O_z$-space and $U \subseteq X$. Then $U$ is a quasi $O_z$-space if $U$ satisfies one of the following:

(a) $U$ is open in $X$

(b) $U$ is dense in $X$

Noting that for a dense subspace $X$ of a space $Y$, $R(X)$ and $R(Y)$ are isomorphic Boolean lattices and that $Z(X)^\#$ is isomorphic with $Z(\beta X)^\#$, one has the following theorem and proposition by the above proposition.

**Theorem 1.8.** For any space $X$, the following are equivalent:

(a) $X$ is a quasi $O_z$-space.

(b) $\nu X$ is a quasi $O_z$-space.

(c) $\beta X$ is a quasi $O_z$-space.

**Remark.** The real line $R$ is an $O_z$-space but $\beta R$ is not an $O_z$-space, which is a quasi $O_z$-space (see [2] and [7]).
Proposition 1.9. For any quasi $O_2$-space $X$, we have:
(a) a $Z^\#$-embedded extension of $X$ is a quasi $O_2$-space;
(b) every regular closed subspace of $X$ is again a quasi $O_2$-space.

The following is due to Dashiell (see [3] for the detail).

Definition 1.10. A covering map $f : X \to Y$ is said to be $Z^\#$-irreducible if $\{ f(A) : A \in Z(X)^\# \} = Z(Y)^\#$.

It is well known that for any covering map $f : X \to Y$, the map $\phi : R(X) \to R(Y)$ ($\phi(A) = f(A)$) is a Boolean algebra isomorphism and the inverse map $\phi^{-1}$ of $\phi$ is given by $\phi^{-1}(B) = cl_X(f^{-1}(int_Y(B)))$. Using this, one has the following.

Theorem 1.11. Suppose $f : X \to Y$ is a covering map and $Y$ is a quasi $O_2$-space, then $f$ is $Z^\#$-irreducible and $X$ is a quasi $O_2$-space.

Proof. Since $f$ is a covering map and $Y$ is a quasi $O_2$-space, $Z(Y)^\# = R(Y) = \{ f(A) : A \in R(X) \}$. Furthermore, for any $B \in Z(Y)^\#$, $cl_X(f^{-1}(int_Y(B))) \in Z(X)^\#$ and $B = f(cl_X(f^{-1}(int_Y(B))))$. Thus $Z(Y)^\# \subseteq \{ f(A) : A \in Z(X)^\# \}$. Since $Z(Y)^\# \subseteq \{ f(A) : A \in Z(X)^\# \} \subseteq \{ f(A) : A \in R(X) \} = Z(Y)^\#$, $\{ f(A) : A \in R(X) \} = \{ f(A) : A \in Z(X)^\# \} = Z(Y)^\#$. So $f$ is $Z^\#$-irreducible. Let $A \in R(X)$. Then there is a $B \in Z(X)^\#$ with $f(A) = f(B)$. Note that $A = cl_X(f^{-1}(int_Y(f(A)))) = cl_X(f^{-1}(int_Y(f(B)))) = B$. Thus $R(X) \subseteq Z(X)^\#$. So $X$ is a quasi $O_2$-space.

Definition 1.12. A space $X$ is said to be:
(a) basically disconnected if every cozero-set in $X$ is $C^*$-embedded in $X$.
(b) quasi F if every dense cozero-set in $X$ is $C^*$-embedded in $X$.
(c) cloz if $B(X) = G(X)$.

Let $B(X)$ denote the set of clopen sets in a space $X$. Then it is known that a space $X$ is basically disconnected iff $B(X) = Z(X)^\#$ (see [8]) and that a space $X$ is a quasi F-space iff for any zero-sets $Z_1$, $Z_2$ with $int_X(Z_1) \cap int_X(Z_2) = \emptyset$, $cl_X(int_X(Z_1)) \cap cl_X(int_X(Z_2)) = \emptyset$. Moreover, $X$ is extremally disconnected iff $R(X) = B(X)$.

Thus one has the following:
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**Proposition 1.13.** For any quasi $O_2$-space $X$, the following are equivalent:

(a) $X$ is a cloz-space.
(b) $X$ is a quasi F-space.
(c) $X$ is a basically disconnected space.
(d) $X$ is an extremally disconnected space.

2. **Quasi $O_2$-Extensions**

Let $\mathcal{C}$ denote the category of completely regular spaces and continuous maps.

**Definition 2.1.** Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}_{\text{Reg}}$ and $X \in \mathcal{C}_{\text{Reg}}$.
(a) A pair $(Y, f)$ is said to be a cover of $X$ if $f : Y \to X$ is a covering map.
(b) A pair $(Y, f)$ is said to be a $\mathcal{C}$-cover of $X$ if $(Y, f)$ is a cover of $X$ and $Y \in \mathcal{C}$.
(c) A $\mathcal{C}$-cover $(Y, f)$ of $X$ is called a minimal $\mathcal{C}$-cover of $X$ if for any $\mathcal{C}$-cover $(Z, g)$ of $X$, there is a covering map $h : Z \to Y$ with $foh = g$.

Let $\mathcal{edc}$, $\mathcal{bdc}$, $\mathcal{QF}$, and $\mathcal{cloz}$ denote the full subcategories of $\mathcal{C}_{\text{Reg}}$ determined by extremally disconnected spaces, basically disconnected spaces, quasi-F spaces, and cloz-spaces, respectively. For any $X \in \mathcal{C}_{\text{Reg}}$, $(E(X), k_X)$, $(\Lambda X, \Lambda_X)$, $(QF(X), \Phi_X)$ and $(E_{cc}(X), z_X)$ denote $\mathcal{edc}$-, $\mathcal{bdc}$-, $\mathcal{QF}$-, and $\mathcal{cloz}$-cover of $X$, respectively (see [3], [4], [5], and [8] for the detail).

The following is immediate from Theorem 1.11 and Proposition 1.13.

**Theorem 2.2.** For any quasi $O_2$-space $X$, $(E(X), k_X)$, $(\Lambda X, \Lambda_X)$, $(QF(X), \Phi_X)$ and $(E_{cc}(X), z_X)$ are equivalent covers of $X$.

**Proposition 2.3.** Consider the following commutative diagram:

\[
P \xrightarrow{f} X \\
\downarrow j_1 \quad \quad \quad \downarrow j_2 \\
Y \xrightarrow{g} W,
\]
where $j_1, j_2$ are dense embeddings and $f, g$ are covering maps. Then $g$ is $Z^#$-irreducible and $j_1$ is $Z^#$-embedded iff $f$ is $Z^#$-irreducible and $j_2$ is $Z^#$-embedded.

Proof. ($\implies$) Take any $A \in Z(P)^#$, then there is a $B \in Z(Y)^#$ with $A = B \cap P$, for $j_1$ is $Z^#$-embedded. Since $f(A) = f(B \cap P) = g(B) \cap X$ and $g$ is $Z^#$-irreducible, $f(A) \in Z(X)^#$. Thus $f$ is $Z^#$-irreducible. Let $C \in Z(X)^#$. Then $cl_P(f^{-1}(int_X(C))) \in Z(P)^#$. Since $j_1$ is $Z^#$-embedded, there is a $D \in Z(Y)^#$ such that $D \cap P = cl_P(f^{-1}(int_X(C)))$. Then $C = f(D \cap P) = g(D) \cap X$. Since $g$ is $Z^#$-irreducible, $g(D) \in Z(X)^#$; therefore $j_2$ is $Z^#$-embedded.

($\impliedby$) Take any $A \in Z(Y)^#$. Then $A \cap P \in Z(P)^#$ for $P$ is dense in $Y$ and $f(A \cap P) = g(A \cap P) = g(A) \cap X$; hence $g(A) \cap X \in Z(X)^#$, because $f$ is $Z^#$-irreducible. Since $j_2$ is $Z^#$-embedded, there is a $B \in Z(W)^#$ with $g(A) \cap X = B \cap X$. Since $j_2$ is a dense embedding and $g(A), B$ are regular closed, $g(A) = B$. Thus $g$ is $Z^#$-irreducible.

Take any $C \in Z(P)^#$, then $f(C) \in Z(X)^#$. Thus there is a $D \in Z(W)^#$ such that $f(C) = D \cap X$. Since $g$ is a covering map, $cl_Y(g^{-1}(int_W(D))) \in Z(Y)^#$. Then $f(cl_Y(g^{-1}(int_W(D))) \cap P) = g(cl_Y(g^{-1}(int_W(D)))) \cap X = D \cap X = f(C)$. Hence $cl_Y(g^{-1}(int_W(D))) \cap P = C$. Thus $j_1$ is $Z^#$-embedded.

We recall that a space $X$ is $Z^#$-embedded in $\beta X$. The following theorem characterizes quasi $O_\omega$-compactifications via $Z^#$-embedding.

**Theorem 2.4.** For any quasi $O_\omega$-space $X$ and any compactification $Y$ of $X$, the following are equivalent:

(a) $j_1 : X \hookrightarrow Y$ is $Z^#$-embedded.
(b) $Y$ is a quasi $O_\omega$-space.
(c) $E(Y) = E_{cc}(Y)$.

Proof. It is known that $\beta E(X) = E(\beta X)$ and the diagram

$$
\begin{array}{c}
E(X) \xrightarrow{k_X} X \\
\beta_{E(X)} \downarrow \quad \downarrow \beta \\
E(\beta X) \xrightarrow{k_{\beta X}} \beta X
\end{array}
$$
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is a pullback.

Consider the pullback diagram

\[
\begin{array}{ccc}
  z_Y^{-1}(X) & \xrightarrow{z_Y^X} & X \\
  j_2 \downarrow & & j_1 \downarrow \\
  E_{cc}(Y) & \xrightarrow{z_Y} & Y.
\end{array}
\]

Since Y is compact, there is a unique continuous map \( f : \beta X \to Y \) with \( f \circ \beta_X = j_1 \). It is easy to show that \( f \) is a covering map; hence there is a covering map \( g : E(\beta X) \to E_{cc}(Y) \) with \( z_Y \circ g = f \circ k_\beta X \).

Thus one has \( j_1 \circ k_X = f \circ \beta_X \circ k_X = f \circ k_\beta X \circ \beta E(X) = z_Y \circ g \circ \beta E(X) \); therefore there is a unique continuous map \( h : E(X) \to z_Y^{-1}(X) \) such that \( z_Y^X \circ h = k_X \) and \( j_2 \circ h = g \circ \beta E(X) \).

(a) \implies (b) It is immediate from Proposition 1.9.

(b) \implies (c) Since X is a quasi O₂-space and \( k_X \) is a covering map, \( k_X \) is \( Z^\# \)-irreducible by Theorem 1.11. Thus \( h \) and \( z_Y^X \) are \( Z^\# \)-irreducible. Since \( j_1 \) is dense, one has a lattice isomorphism between \( R(Y) \) and \( R(X) = Z(Y)^\# \) via \( A \mapsto A \cap X \) (\( A \in R(Y) \)), so that \( j_1 \) is \( Z^\# \)-embedded; therefore \( j_2 \) is also \( Z^\# \)-embedded, because of the above proposition. Thus \( z_Y^{-1}(X) \) is a cloz-space; hence an extremally disconnected space and \( \beta z_Y^{-1}(X) = E_{cc}(Y) \). Since \( h \) is a covering map, \( h \) is a homeomorphism. Hence \( \beta E(X) = E(\beta X) = \beta z_Y^{-1}(X) = E_{cc}(Y) \). Since \( E(Y) = E(\beta X) \), \( E(Y) = E_{cc}(Y) \).

(c) \implies (a) Since \( E_{cc}(Y) \) is an extremally disconnected space, \( g \) is a homeomorphism. Hence \( z_Y \circ g = f \circ k_\beta X \) is \( Z^\# \)-irreducible. Since \( \beta E(X) \) is \( Z^\# \)-embedded, by Proposition 2.3, \( j_1 \) is \( Z^\# \)-embedded.

References

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