ON UNIFORMLY ULTRA SEPARATING FAMILY OF FUNCTION ALGEBRAS

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1. Introduction

Let $X$ be a compact Hausdorff space, and let $C(X)$ (resp. $C_{\mathbb{R}}(X)$) be the complex (resp. real) Banach algebra of all continuous complex-valued (resp. real-valued) functions on $X$ with the pointwise operations and the supremum norm $\| \cdot \|_X$. A Banach function algebra on $X$ is a Banach algebra lying in $C(X)$ which separates the points of $X$ and contains the constants. A Banach function algebra on $X$ equipped with the supremum norm is called a uniform algebra on $X$, that is, a uniformly closed subalgebra of $C(X)$ which separates the points of $X$ and contains the constants.

Let $E$ be a (real or complex) normed linear space with norm $\| \cdot \|_E$. Denote by $\hat{E} = \ell^\infty(N, E)$ the space of all bounded functions from the set $N = \{1, 2, 3, \ldots\}$ to $E$ normed as follows:

$$\| \hat{f} \|_{\hat{E}} = \sup\{\| f_n \|_E : n \in N\} < \infty$$

for a sequence $\hat{f} = \{ f_n \}_{n=1}^\infty$ in $\hat{E}$.

Denote by $\check{X} = \beta(N \times X)$ the Stone-Čech compactification of the product space $N \times X$. Since every sequence $\{ f_n \}_{n=1}^\infty$ in $\ell^\infty(N, C(X))$ can be considered as a function from $N \times X$ to $C$, it has a unique continuous extension to a function in $C(\check{X})$. So, we have $\ell^\infty(N, C(X)) = C(\check{X})$.

**Definition 1.1 ([2]).** Let $E$ be a (real or complex) normed linear space continuously injected in $C(X)$. We say that $E$ is ultraseparating on $X$ if $\hat{E}$ separates the points of $\check{X}$.

Let $A$ be a Banach function algebra on $X$. Denote by $\text{ball}A$ the set of all functions in $A$ with $\| f \|_A \leq 1$.

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PROPOSITION 1.2 ([1]). Let $A$ be a Banach function algebra on $X$. Then the following are equivalent:

1. $A$ is ultraseparating on $X$.
2. There exist a $\delta > 0$ and an $n \in \mathbb{N}$ such that for any pair of non-empty disjoint closed subsets $E$ and $F$ of $X$, there exist $u_1, \ldots, u_n; v_1, \ldots, v_n \in \text{ball}A$ such that

$$\sum_{i=1}^{n} (|u_i| - |v_i|) \geq \delta \quad \text{on } E,$$

$$\sum_{i=1}^{n} (|u_i| - |v_i|) \leq -\delta \quad \text{on } F.$$

Notice that if $A$ is ultraseparating for some $\delta > 0$ and $n \in \mathbb{N}$, then $A$ is ultraseparating for any positive integer $m \geq n$ with same $\delta > 0$. So, we may assume that $\delta \geq 1/n$, and hence we can redefine ultraseparability of a Banach function algebra as follows:

DEFINITION 1.3. A Banach function algebra $A$ on $X$ is said to be $n$-ultraseparating on $X$ for some $n \in \mathbb{N}$ if for any pair of non-empty disjoint closed subsets $E$ and $F$ of $X$, there exist $u_1, \ldots, u_n; v_1, \ldots, v_n \in \text{ball}A$ such that

$$\sum_{i=1}^{n} (|u_i| - |v_i|) \geq 1/n \quad \text{on } E,$$

$$\sum_{i=1}^{n} (|u_i| - |v_i|) \leq -1/n \quad \text{on } F.$$

DEFINITION 1.4. Let $\Gamma$ be an index set, and let $A_\gamma$ be an $n_\gamma$-ultraseparating Banach function algebras on $X_\gamma$ for $\gamma \in \Gamma$. The family \{\(A_\gamma : \gamma \in \Gamma\}\} is said to be uniformly ultraseparating if $\sup\{n_\gamma : \gamma \in \Gamma\} < \infty$.

Now, let $A_\gamma$ be a Banach function algebra on $X_\gamma$ for each $\gamma \in \Gamma$, and let $X = \bigcup_{\gamma \in \Gamma} (\{\gamma\} \times X_\gamma)$. Define an indexed family $\tilde{f} = \{f_\gamma\}_{\gamma \in \Gamma}$
of functions $f_\gamma \in A_\gamma$ by

$$\tilde{f}(\gamma, x_\gamma) = f_\gamma(x_\gamma) \text{ for } (\gamma, x_\gamma) \in \{\gamma\} \times X_\gamma.$$  

Let $\tilde{A}$ be the set of all $\tilde{f} = \{f_\gamma\}_{\gamma \in \Gamma}$ such that $\sup\{\|f_\gamma\|_{A_\gamma} : \gamma \in \Gamma\} < \infty$. Then $\tilde{A}$ is a Banach algebra (with the norm $\|\tilde{f}\|_{\tilde{A}} = \sup\{\|f_\gamma\|_{A_\gamma} : \gamma \in \Gamma\}$) lying in $C(\beta X)$ which contains the constants. Indeed, $\|\tilde{f}\|_{\tilde{A}} \geq \|\tilde{f}\|_{\beta X}$ because $\|\tilde{f}\|_{\beta X} = \sup\{\|f_\gamma\|_{X_\gamma} : \gamma \in \Gamma\}$.

In this paper, we will study point separability and ultraseparability of the algebra $\tilde{A}$ via uniform ultraseparability of the family $\{A_\gamma : \gamma \in \Gamma\}$.

2. Main Results

**Lemma 2.1** ([3]). Let $A$ be a real Banach function algebra on $X$. Then the space of all linear combinations of $|f|$ for $f \in \text{ball}A$ is uniformly dense in $C_\mathbb{R}(X)$.

**Theorem 2.2.** Let $A_\gamma$ be a Banach function algebra on $X_\gamma$ for each $\gamma \in \Gamma$. Then the following are equivalent:

1. $\tilde{A}$ separates the points of $\beta X$.
2. There exists a finite subset $\Gamma_0$ of $\Gamma$ such that the family $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating.

**Proof.** (2) $\implies$ (1): Let $\Gamma_0$ be a finite subset of $\Gamma$ such that the family $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating. Let $p$ and $q$ be distinct two points of $\beta X$. Put

$$Y_1 = \bigcup_{\gamma \in \Gamma_0} (\{\gamma\} \times X_\gamma) \text{ and } Y_2 = X - Y_1.$$  

Then $\beta X = \beta(Y_1 \cup Y_2) = \beta Y_1 \cup \beta Y_2 = Y_1 \cup \beta Y_2$ since $\Gamma_0$ is a finite set.

**Case 1.** $p$ and $q \in Y_1$:
If $p, q \in \{\gamma_0\} \times X_{\gamma_0}$ for some $\gamma_0 \in \Gamma_0$, then $p = (\gamma_0, x), q = (\gamma_0, y)$ for some distinct $x, y \in X_{\gamma_0}$. Since $A_{\gamma_0}$ separates the points of $X_{\gamma_0}$, there
exists \( g \in A_{\gamma_0} \) such that \( g(x) \neq g(y) \). Choose any \( \tilde{f} = \{ f_{\gamma} \}_{\gamma \in \Gamma} \in \tilde{A} \) such that \( f_{\gamma_0} = g \). Then \( \tilde{f}|_{\{ \gamma_0 \} \times X_{\gamma_0}} = f_{\gamma_0} \), and

\[
\tilde{f}(p) = f_{\gamma_0}(x) \neq f_{\gamma_0}(y) = \tilde{f}(q).
\]

Next, suppose that \( p = (\gamma_0, x_{\gamma_0}) \in \{ \gamma_0 \} \times X_{\gamma_0}, q = (\gamma_1, x_{\gamma_1}) \in \{ \gamma_1 \} \times X_{\gamma_1} \) for some distinct \( \gamma_0, \gamma_1 \in \Gamma_0 \). Define for \( \gamma \in \Gamma \),

\[
f_{\gamma} = \begin{cases} 
0 & \text{if } \gamma = \gamma_0, \\
1 & \text{if } \gamma \neq \gamma_0.
\end{cases}
\]

Then \( \tilde{f} = \{ f_{\gamma} \}_{\gamma \in \Gamma} \in \tilde{A} \), and

\[
\tilde{f}(p) = f_{\gamma_0}(x_{\gamma_0}) = 0 \neq 1 = f_{\gamma_1}(x_{\gamma_1}) = \tilde{f}(q).
\]

**Case 2.** \( p \in Y_1, q \in \beta Y_2 \) (or \( p \in \beta Y_2, q \in Y_1 \)):

Assume that \( p \in Y_1, q \in \beta Y_2 \). Then \( p = (\gamma_0, x_0) \) for some \( \gamma_0 \in \Gamma_0 \) and \( x_0 \in X_{\gamma_0} \). Define

\[
f_{\gamma} = \begin{cases} 
0 & \text{if } \gamma \in \Gamma - \Gamma_0, \\
1 & \text{if } \gamma \in \Gamma_0,
\end{cases}
\]

and let \( \tilde{f} = \{ f_{\gamma} \}_{\gamma \in \Gamma} \). Then \( \tilde{f} \in \tilde{A}, \tilde{f}|_{\beta Y_2} = 0 \), and

\[
\tilde{f}(p) = f_{\gamma_0}(x_0) = 1 \neq 0 = \tilde{f}(q).
\]

**Case 3.** \( p, q \in \beta Y_2 \):

It suffices to show that \( \tilde{A}|_{\beta Y_2} \) separates the points of \( \beta Y_2 \). Indeed, if then, we can choose \( \tilde{g} = \{ g_{\gamma} \}_{\gamma \in \Gamma - \Gamma_0} \in \tilde{A}|_{\beta Y_2} \) such that \( \tilde{g}(p) \neq \tilde{g}(q) \).

Define

\[
f_{\gamma} = \begin{cases} 
g_{\gamma} & \text{if } \gamma \in \Gamma - \Gamma_0, \\
0 & \text{if } \gamma \in \Gamma_0,
\end{cases}
\]

and let \( \tilde{f} = \{ f_{\gamma} \}_{\gamma \in \Gamma} \). Then \( \tilde{f} \in \tilde{A} \) and \( \tilde{f}|_{\beta Y_2} = \tilde{g} \). Hence, we have

\[
\tilde{f}(p) = \tilde{g}(p) \neq \tilde{g}(q) = \tilde{f}(q).
\]
Now, to prove that $\tilde{A}|_{\beta Y_2}$ separates the points of $\beta Y_2$, let $p$ and $q$ be distinct two points of $\beta Y_2$, and let $U$ and $V$ be neighborhoods of $p$ and $q$, respectively in $\beta Y_2$ having disjoint closures. Put

$$U^\gamma = (\{\gamma\} \times X_\gamma) \cap U, \quad V^\gamma = (\{\gamma\} \times X_\gamma) \cap V$$

for $\gamma \in \Gamma - \Gamma_0$. (Here, $\overline{S}$ is the closure of $S$ in $\beta Y_2$.) Then $U^\gamma$ and $V^\gamma$ are disjoint closed sets in $\{\gamma\} \times X_\gamma$ which is homeomorphic to $X_\gamma$ for each $\gamma \in \Gamma - \Gamma_0$. Since $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating, we can choose $n \in \mathbb{N}$ such that for each $\gamma \in \Gamma - \Gamma_0$, there exist $u_1^\gamma, \ldots, u_n^\gamma, v_1^\gamma, \ldots, v_n^\gamma \in \text{ball}(A_\gamma)$ such that

$$\sum_{i=1}^{n} (|u_i^\gamma| - |v_i^\gamma|) \geq 1/n \quad \text{on} \quad U^\gamma,$$

$$\sum_{i=1}^{n} (|u_i^\gamma| - |v_i^\gamma|) \leq -1/n \quad \text{on} \quad V^\gamma.$$

Put $\tilde{u}_i = \{u_i^\gamma\}_{\gamma \in \Gamma - \Gamma_0}, \tilde{v}_i = \{v_i^\gamma\}_{\gamma \in \Gamma - \Gamma_0}$ for $i = 1, \ldots, n$. Then $\tilde{u}_i, \tilde{v}_i \in \text{ball}(\tilde{A}|_{\beta Y_2})$ for $i = 1, \ldots, n$, and

\[
\begin{align*}
\sum_{i=1}^{n} (|\tilde{u}_i| - |\tilde{v}_i|) & \geq 1/n \quad \text{on} \quad \overline{U}, \\
\sum_{i=1}^{n} (|\tilde{u}_i| - |\tilde{v}_i|) & \leq -1/n \quad \text{on} \quad \overline{V}
\end{align*}
\]

because $X \cap U$ and $X \cap V$ are dense in $\overline{U}$ and $\overline{V}$, respectively.

Now, if $\tilde{f}(p) = \tilde{f}(q)$ for all $\tilde{f} \in \tilde{A}|_{\beta Y_2}$, then (*) is impossible. Therefore, $\tilde{A}|_{\beta Y_2}$ separates the points of $\beta Y_2$.

(1) $\implies$ (2): Suppose that $\tilde{A}$ separates the points of $\beta X$, and suppose that there is no finite subset of $\Gamma_0$ of $\Gamma$ such that $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating. For each $n \in \mathbb{N}$, put

$$\Gamma_n = \{\gamma \in \Gamma : A_\gamma \text{ is not } n\text{-ultraseparating}\}.$$
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Then \( \Gamma_n \) must be an infinite set for each \( n \in \mathbb{N} \), and \( \Gamma_1 \supset \Gamma_2 \supset \ldots \). Take \( \gamma_1, \gamma_2, \ldots \in \Gamma \) distinct so that \( \gamma_n \in \Gamma_n \). Then for each \( n \in \mathbb{N} \), we can choose disjoint closed sets \( E_n \) and \( F_n \) in \( X_{\gamma_n} \) such that for any \( f_1, \ldots, f_n; g_1, \ldots, g_n \in \text{ball} \, \mathcal{A}_{\gamma_n} \), either

\[
\sum_{i=1}^{n} (|f_i| - |g_i|) < 1/n \quad \text{on} \quad E_n
\]

or

\[
\sum_{i=1}^{n} (|f_i| - |g_i|) > -1/n \quad \text{on} \quad F_n.
\]

Let \( E = \bigcup_{n=1}^{\infty} (\{\gamma_n\} \times E_n) \) and \( F = \bigcup_{n=1}^{\infty} (\{\gamma_n\} \times F_n) \). (Here, \( \overline{S} \) is the closure of \( S \) in \( \beta X \).) Then \( E \) and \( F \) are disjoint closed sets in \( \beta X \). Let \( \tilde{h} \) be a real-valued continuous function from \( \beta X \) onto \([-1, 1]\) such that \( \tilde{h}|_E = 1, \tilde{h}|_F = -1 \). Since \( \tilde{A} \) separates the point of \( \beta X \), so does \( \text{Re} \tilde{A} \). Hence by the above Lemma 2.1, there are a positive integer \( N \) and \( \tilde{u}_i = \{u_i^\gamma\}_{\gamma \in \Gamma}, \tilde{v}_i = \{v_i^\gamma\}_{\gamma \in \Gamma} \in \tilde{A} \) such that

\[
\left| \sum_{i=1}^{N} (|\tilde{u}_i| - |\tilde{v}_i|) - \tilde{h} \right| \leq 1/2 \quad \text{on} \quad \beta X.
\]

So, for any \( n \in \mathbb{N} \), we have

\[
\left| \sum_{i=1}^{N} (|u_i^\gamma^n| - |v_i^\gamma^n|) - 1 \right| \leq 1/2 \quad \text{on} \quad E_n
\]

and

\[
\left| \sum_{i=1}^{N} (|u_i^\gamma^n| - |v_i^\gamma^n|) + 1 \right| \leq 1/2 \quad \text{on} \quad F_n.
\]
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Take \( c = \max_{1 \leq i \leq N} \{ \| \hat{u}_i \|, \| \hat{v}_i \| \} \). Then for \( n \in \mathbb{N} \) with \( 1/2c \geq 1/n \), we have

\[
\sum_{i=1}^{N} (|f_i^u| - |g_i^u|) \geq 1/n \quad \text{on } E_n
\]

and

\[
\sum_{i=1}^{N} (|f_i^v| - |g_i^v|) \leq -1/n \quad \text{on } F_n,
\]

where \( f_i^u = (1/c)u_i^\gamma \) and \( g_i^u = (1/c)v_i^\gamma \) for \( \gamma \in \Gamma \). But this is impossible. \( \square \)

**Theorem 2.3.** Let \( A_\gamma \) be a Banach function algebra on \( X_\gamma \) for each \( \gamma \in \Gamma \). Then the following are equivalent:

1. The family \( \{ A_\gamma : \gamma \in \Gamma \} \) is uniformly ultraseparating.
2. \( \hat{A} \) is an ultraseparating Banach function algebra on \( \beta X \).

**Proof.** (1) \( \implies \) (2): Suppose that \( \{ A_\gamma : \gamma \in \Gamma \} \) is uniformly ultraseparating. Then there exists \( n \in \mathbb{N} \) such that each \( A_\gamma \) is \( n \)-ultraseparating on \( X_\gamma \). Let \( E \) and \( F \) be disjoint closed subsets of \( \beta X \) with non-empty interior, and put

\[
E^\gamma = (\{ \gamma \} \times X_\gamma) \cap E, \quad F^\gamma = (\{ \gamma \} \times X_\gamma) \cap F
\]

for each \( \gamma \in \Gamma \). Then \( E^\gamma \) and \( F^\gamma \) are non-empty disjoint closed sets in \( \{ \gamma \} \times X_\gamma \) for each \( \gamma \in \Gamma \). So, for each \( \gamma \in \Gamma \), we can choose \( u_1^\gamma, \ldots, u_n^\gamma, v_1^\gamma, \ldots, v_n^\gamma \in \text{ball}A_\gamma \) such that

\[
\sum_{i=1}^{n} (|u_i^\gamma| - |v_i^\gamma|) \geq 1/n \quad \text{on } E^\gamma,
\]

\[
\sum_{i=1}^{n} (|u_i^\gamma| - |v_i^\gamma|) \leq -1/n \quad \text{on } F^\gamma.
\]
Take \( \tilde{u}_i = \{u_i^\gamma\}_{\gamma \in \Gamma}, \tilde{v}_i = \{v_i^\gamma\}_{\gamma \in \Gamma} \) for \( i = 1, \ldots, n \). Then \( \tilde{u}_i, \tilde{v}_i \in \text{ball} \tilde{A} \) for \( i = 1, \ldots, n \), and

\[
\sum_{i=1}^n (|\tilde{u}_i| - |\tilde{v}_i|) \geq 1/n \quad \text{on } E,
\]

\[
\sum_{i=1}^n (|\tilde{u}_i| - |\tilde{v}_i|) \leq -1/n \quad \text{on } F.
\]

Therefore, \( \tilde{A} \) is \( n \)-ultraseparating on \( \beta X \).

(2) \( \implies \) (1): Suppose \( \tilde{A} \) is \( n \)-ultraseparating on \( \beta X \), and fix \( \gamma_0 \in \Gamma_0 \). We will show that \( A_{\gamma_0} = \tilde{A}|\{\gamma_0\} \times X_{\gamma_0} \) is \( n \)-ultraseparating on \( X_{\gamma_0} \).

Let \( E \) and \( F \) be disjoint closed subsets of \( X_{\gamma_0} \). Then \( \{\gamma_0\} \times E \) and \( \{\gamma_0\} \times F \) are disjoint closed sets in \( \beta X \). So, there exist \( \tilde{u}_1, \ldots, \tilde{u}_n; \tilde{v}_1, \ldots, \tilde{v}_n \in \text{ball} \tilde{A} \) such that

\[
\sum_{i=1}^n (|\tilde{u}_i| - |\tilde{v}_i|) \geq 1/n \quad \text{on } \{\gamma_0\} \times E,
\]

\[
\sum_{i=1}^n (|\tilde{u}_i| - |\tilde{v}_i|) \leq -1/n \quad \text{on } \{\gamma_0\} \times F.
\]

Denote \( \tilde{u}_i = \{u_i^\gamma\}_{\gamma \in \Gamma}, \tilde{v}_i = \{v_i^\gamma\}_{\gamma \in \Gamma} \) for \( u_i^\gamma, v_i^\gamma \in A_\gamma, \gamma \in \Gamma, i = 1, \ldots, n \). Then \( u_i^{\gamma_0} \) and \( v_i^{\gamma_0} \in \text{ball} A_{\gamma_0} \) for \( i = 1, \ldots, n \), and

\[
\sum_{i=1}^n (|u_i^{\gamma_0}| - |v_i^{\gamma_0}|) \geq 1/n \quad \text{on } E,
\]

\[
\sum_{i=1}^n (|u_i^{\gamma_0}| - |v_i^{\gamma_0}|) \leq -1/n \quad \text{on } F.
\]

Thus, \( A_{\gamma_0} \) is \( n \)-ultraseparating on \( X_{\gamma_0} \), and therefore \( \{A_\gamma: \gamma \in \Gamma\} \) is uniformly ultraseparating. \( \Box \)

Let \( B \) be a Banach function algebra on a compact Hausdorff space \( Y \). In the notations of the above theorem, take \( \Gamma = \mathbb{N}, A_\gamma = B, \) and \( X_\gamma = Y \) for every \( \gamma \in \Gamma \). Then \( X = \bigcup_{n \in \mathbb{N}} \{\{n\} \times X_n\} = \mathbb{N} \times Y, \) so \( \beta X = \beta(\mathbb{N} \times Y) = \tilde{Y} \) and \( \tilde{A} = \ell^\infty(\mathbb{N}, B) = \tilde{B} \). Thus, we have

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Corollary 2.4 ([1]). Let \( B \) be a Banach function algebra on a compact Hausdorff space \( Y \). Then the following are equivalent:

1. \( B \) is ultraseparating on \( Y \).
2. \( \hat{B} \) is ultraseparating on \( \hat{Y} \).

Example 2.5. Let \( X = [0, 1] \). For each \( n \in \mathbb{N} \), define

\[
\|f\|_n = \|f\|_X + n|f(0)| \quad \text{for} \quad f \in C(X),
\]

where \( \| \|_X \) is the supremum norm. Then \( C(X) \) equipped with the norm \( \| \|_n \) is a Banach function algebra on \( X \), which we denote by \( \mathcal{A}_n \) for \( n \in \mathbb{N} \). Since \( \|f\|_X \leq \|f\|_n \leq (n + 1)\|f\|_X \) for \( f \in C(X) \) and since the uniform algebra \( C(X) \) is 1-ultraseparating, \( \mathcal{A}_n \) is \((n + 1)\)-ultraseparating for each \( n \in \mathbb{N} \).

Let \( \mathcal{A} \) be the Banach algebra so defined as in Definition 1.4, and let \( C(\hat{X}) = \ell^\infty(N, C(X)) \) with the uniform norm on \( \hat{X} = \beta(N \times X) \). Then, we have

\[
\tilde{f} = \{f_n\}_{n=1}^\infty \in \mathcal{A} \iff \tilde{f} \in C(\hat{X}) \quad \text{and} \quad \sup_{n \in \mathbb{N}} n|f_n(0)| < \infty.
\]

Hence \( \mathcal{A} \) is a proper subalgebra of \( C(\hat{X}) \), and therefore by Theorem 2.3 the family \( \{\mathcal{A}_n : n \in \mathbb{N} \} \) is not uniformly ultraseparating. Actually, we can show this fact directly as follows:

Let \( k \) be any positive integer less than \( \sqrt{n + 1} \). Then \( |f(0)| \leq 1/(n + 1) \) for \( f \in \text{ball}\mathcal{A}_n \). Take \( E = \{0\}, F = \{1\} \). If \( f_1, \ldots, f_k, g_1, \ldots, g_k \in \text{ball}\mathcal{A}_n \), then we have

\[
\sum_{i=1}^k (|f_i(0)| - |g_i(0)|) \leq \sum_{i=1}^k |f_i(0)| \leq \frac{k}{n + 1} < \frac{\sqrt{n + 1}}{n + 1} = \frac{1}{\sqrt{n + 1}} < \frac{1}{k}.
\]

Thus, \( \mathcal{A}_n \) is not \( k \)-ultraseparating, and therefore \( \mathcal{A}_n \) is \( k_n \)-ultraseparating for a positive integer \( k_n \geq \sqrt{n + 1} \) for each \( n \in \mathbb{N} \). This implies that the family \( \{\mathcal{A}_n : n \in \mathbb{N} \} \) is not uniformly ultraseparating because \( k_n \to \infty \) as \( n \to \infty \).
Remark. Let $A_\gamma = C(X_\gamma)$ with the uniform norm on $X_\gamma$ for each $\gamma \in \Gamma$. Then $\tilde{A}$ is an ultraseparating uniform algebra on $\beta X$ by Theorem 2.3 and by the fact that $\|\hat{f}\|_{\tilde{A}} = \|\hat{f}\|_{\beta X}$ for $\hat{f} \in \tilde{A}$. Since each $A_\gamma$ is self-adjoint, so is $\tilde{A}$, and therefore $\tilde{A} = C(\beta X)$ by the Stone-Weierstrass Theorem.

But this is not true if $A_\gamma = C(X_\gamma)$ is equipped with any other norm than the supremum norm as in the above example.

References