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EXACT SEQUENCES FOR SUMS OF PAIRWISE I.I.D. RANDOM VARIABLES

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Let $\{X_n\}$ be a sequence of random variables and $S_n = \sum_{i=1}^n X_i$. Chow and Robbins [1] proved that if $\{X_n\}$ is a sequence of independent, identically distributed random variables with $E|X_1| = \infty$, then for any sequence of constant b_n,$P(lim_{n\to\infty}\frac{S_n}{b_n} = 1) = 0$ using Theorem 2 of Feller [3]. In this paper, only by requiring the random variables to be pairwise independent and identically distributed, we obtain the same result without using Kolmogorov's theorem by which Feller's theorem is reduced essentially.

In this paper, $X, X_1, X_2, \cdots$ will denote any sequence of pairwise independent random variables with common distribution, and $b_1, b_2 \cdots$ will denote any sequence of constants. Using Chung [2,Theorem 4.2.5] and the same idea as in Chow and Robbins [1, Lemma 1 and 2] we have the following lemma.

LEMMA 1. *If b_n/n is positive and non-decreasing, then*

$$\sum_1^\infty P(|X| > b_n) = \infty \text{ implies } P(\overline{lim}\frac{|X_n|}{b_n} = \infty)$$
$$= P(\overline{lim}\frac{|S_n|}{b_n} = \infty) = 1.$$

Now we consider the following lemma.

LEMMA 2. *If b_n/n is positive and non-decreasing and if, in addition, $E|X| = \infty$, then for any subsequence b_{n_k},*

$$\sum_1^\infty P(|X| > b_n) < \infty \text{ implies } P(\underline{lim}\frac{|S_{n_k}|}{b_{n_k}} = 0) = 1.$$

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Firstly, suppose that

$$\overline{lim}\frac{|b_n|}{n} < \infty.$$

Then from (2), $P(\overline{lim}\frac{|S_n|}{n} < \infty) > 0$, and hence by Lemma 1 for $b_n = n$, $\sum P(|X| > n) < \infty$ and hence $E|X| < \infty$, which contadicts the hypothesis.
Alternatively, suppose that

$$\overline{lim}\frac{|b_n|}{n} = \infty.$$

Set

(3) $$\alpha_n = n\ max\left[\frac{|b_1|}{1}, \frac{|b_2|}{2}, \cdots, \frac{|b_n|}{n}\right] \geq |b_n| \geq 0.$$

Then the sequence α_n/n is positive and non–decreasing and there exists a sequence of integer $n_1 < n_2 < \cdots$ such that

(4) $$\alpha_{n_k} = |b_{n_k}|.$$

From (2) and (3), it follows that

$$P(\overline{lim}\frac{|S_n|}{\alpha_n} < \infty) > 0,$$

and hence by Lemma 1 and 2,

(5) $$P(\underline{lim}\frac{|S_{n_k}|}{\alpha_{n_k}} = 0) = 1.$$

From (4) and (5), it follows that

$$P(\underline{lim}\left|\frac{S_n}{b_n}\right| = 0) = 1$$

which contradicts the first half of (2) and the proof of the theorem is complete.