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AUTOMORPHISM GROUPS ON CERTAIN REINHARDT DOMAINS

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1. Introduction and Statements of Results

In relation to the characterization of domains in $\mathbb{C}^n$ the study of automorphism groups on domains in $\mathbb{C}^n$ is attracting much attention lately. For example, Wong proved that any bounded strongly pseudo-convex domain in $\mathbb{C}^n$ with noncompact automorphism group is biholomorphically equivalent to the unit ball in $\mathbb{C}^n$ [10]. Rosay also proved a more general version of the same theorem [8]. As a generalization of Wong-Rosay's theorem in $\mathbb{C}^2$, Bedford and Pinchuk proved that any bounded pseudoconvex domain with real analytic boundary is biholomorphically equivalent to a domain of the form

$$E_m = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1 \} \tag{1.1}$$

for some positive integer m provided that the automorphism group of the domain is noncompact. Kim also proved a similar result [6]. On the other hand, Greene and Krantz conjectured that the only domain in $\mathbb{C}^2$ with noncompact automorphism group is E_m [5, see also 2].

In this paper, we show that Greene-Krantz's conjecture is true for certain class of domains. In fact, we give a complete classification of automorphism groups of domains of the form

$$E_\phi = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + \phi(|z_2|^2) < 1 \} \tag{1.2}$$

where the function ϕ is a real valued C^∞ function in a neighborhood of $[0,1]$ which satisfies the following conditions:

(1) $\phi(0) = \phi'(0) = 0$ and $\phi(1) = 1$,
(2) $\phi(t)$ is increasing and convex for $t > 0$

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PROPOSITION 3. *Let $\psi(t)$ be a real valued C^3 function for $t > 0$ and let m be a Möbius transform on the unit disc Δ in $\mathbb{C}$ such that $m(0) \neq 0$. If the function $u(z) = \psi(1 - |m(z)|^2) - \psi(1 - |z|^2)$ is harmonic in Δ, then*

$$\psi(t) = A \log t + B$$

for some constant A and B.

Note that if $\psi(t) = A \log t + B$, then $\psi(1 - |m(z)|^2) - \psi(1 - |z|^2)$ is harmonic in Δ.

At the Symposium on Complex Analysis at Madison in honor of Professor Walter Rudin where this result was announced, Steven Krantz informed me that he and Greene had a result which had some intersactions with our result [11].

2. Proofs

As before, we let ϕ be a real valued C^∞ function in a neighborhood of $[0, 1]$ which satisfies the following conditions;

(1) $\phi(0) = \phi'(0) = 0$ and $\phi(1) = 1$,
(2) $\phi(t)$ is increasing and convex for $t > 0$.

And let

$$E_\phi = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + \phi(|z_2|^2) < 1 \}.$$

Then the Levi form on E_ϕ is given by

$$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & \phi''(|z_2|^2)|z_2|^2 + \phi'(|z_2|^2) \end{pmatrix}.$$

Hence E_ϕ is a pseudo-convex domain; weakly pseudo-convex along $(e^{i\theta}, 0) \in \partial\Omega_\phi$ and strongly pseudo-convex otherwise. We let $W = \{(e^{i\theta}, 0) \; ; \; |\theta| \leq \pi\}$. The principal observation in this paper is that any automorphism F on E_ϕ maps the set W onto itself. In fact, since E_ϕ is a Reinhardt domain, the Bergmann projection on E_ϕ maps $C^\infty(\overline{E_\phi})$ into $C^\infty(\overline{E_\phi})$ [main theorem in 4]. So, by well known Bell-Ligocka's theorem, any automorphism on E_ϕ can be extended as a diffeomorphism on $\overline{E_\phi}$. We call the extended function F. Since the type of the boundary is a biholomorphic invariant, we have $F(W) = W$.

By letting $z_2 = 0$, one can easily see that $a = 1$ and $c = 0$. We observe that $G(0,0) = 0$. We also note, by computing the Jacobian of F for $z_2 = 0$, that $H(z_1, 0) \neq 0$. It then follows that

$$\lambda G(z_1, \lambda z_2) = G(z_1, z_2) + bH(z_1, z_2) \tag{4}$$

$$H(z_1, \lambda z_2) = H(z_1, z_2) \tag{5}$$

for any λ with $|\lambda| = 1$. (5) implies that $H(z_1, z_2)$ is independent of z_2. And (4) implies that $\lambda G(z_1, \lambda z_2) = G(z_1, z_2)$ for any λ with $|\lambda| = 1$ and hence $G \equiv 0$. Let $h(z_1) = H(z_1, z_2)$.

So far, we proved that $f_1(z_1, z_2) = m(z_1)$ and $f_2(z_1, z_2) = z_2 h(z_1)$ if $|z_1| < \delta$ and $|z_2| < \delta$. It immediately follows that $f_1(z_1, z_2) = m(z_1)$ on E_ϕ and that $h(z_1)$ can be extended as a holomorphic function in Δ. This completes the proof. □

Proof of Proposition 3. Let

$$m(z) = \lambda \frac{z - \beta}{1 - \overline{\beta} z}.$$

Then, $\beta \neq 0$. We may assume that β is real since the Laplacian is rotation invariant. Since the function $u(z) = \psi(1 - |m(z)|^2) - \psi(1 - |z|^2)$ is harmonic in Δ, we have

$$\begin{aligned}\psi''(1 - |m(z)|^2)|m'(z)|^2|m(z)|^2 - \psi'(1 - |m(z)|^2)|m'(z)|^2 \\ - \psi''(1 - |z|^2)|z|^2 + \psi'(1 - |z|^2) = 0\end{aligned}$$

If we let

$$v(z) = \psi''(1 - |m(z)|^2)|m'(z)|^2|m(z)|^2 - \psi'(1 - |m(z)|^2)|m'(z)|^2,$$

then v is radial. Hence $v_\theta \equiv 0$. A complicated but straight forward computation shows that

$$\begin{aligned}\frac{(1 + \beta^2 r^2)^3}{2\beta r(1 - \beta^2)} v_\theta(re^{i\frac{\pi}{2}}) &= (t^2 - t)\psi'''(t) + 2(t - 1)\psi''(t) + 2t\psi''(t) + 2\psi'(t) \\ &= 0\end{aligned}$$

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