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PATTERSON–SULLIVAN MEASURE AND GROUPS OF DIVERGENCE TYPE

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1. Introduction

For a nonelementary group Γ of hyperbolic isometries acting on B^{d+1}, the critical exponent $\delta(\Gamma)$ is defined as

$$\delta(\Gamma) = \inf \left\{ \alpha : \sum_{\gamma \in \Gamma} e^{-\alpha(0,\gamma(0))} < \infty \right\}$$

where $(0, \gamma(0))$ is the hyperbolic distance from 0 to $\gamma(0)$. The group Γ is said to be of convergence type according as the series

$$\sum_{\gamma \in \Gamma} e^{-\delta(\Gamma)(0,\gamma(0))}$$

converges or diverges.

A point $\xi \in S^d$ is a limit point for the discrete group Γ if for one, and hence every, point $x \in B^{d+1}$ the orbit $\Gamma(x)$ accumulate at ξ. The set of limit points is denoted by $\Lambda(\Gamma)$, or simply Λ. The point $\xi \in \Lambda(\Gamma)$ is said to be a conical limit point (approximation point) for Γ if for every $a \in B^{d+1}$ there exists a sequence $\{\gamma_n\} \subset \Gamma$ on which the sequence $\frac{|\xi - \gamma_n(a)|}{1-|\gamma_n(a)|}$ remains bounded. The set of conical limit points is denoted by Λ_c.

In this paper, we use the Patterson-Sullivan measure and results of [**H**] to show that for a nonelementary discrete group of divergence type, the conical limit set Λ_c has positive Patterson-Sullivan measure. The definition of the Patterson-Sullivan measure for groups of divergence

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type is reviewed in section 2. The Patterson-Sullivan measure can also be defined for groups of convergence type and the details for that case can be found in [N]. Necessary definitions and results from [H] are given in section 3, and in section 4, we prove our main result.

To state our result, we must define the shadow and projection of a point $a \in B^{d+1}$. Denote by $B_\rho(a)$ the ball of center a and non-Euclidean radius ρ for $\rho > 0$, and by B_ρ the ball of radius ρ at 0. Given a point $a \in B^{d+1}$, $a \neq o$, define the projection $\mathrm{pr}\{a\}$ in S^d to be $\frac{a}{|a|}$ and the shadow $\mathrm{sh}\{a\}$ in $B^{d+1} \cup S^d$ to be the closed line segment $[a, \mathrm{pr}\{a\}]$. For a set $E \subseteq B^{d+1}$ which does not contain 0, we define $\mathrm{pr}E = \bigcup_{x \in E} \mathrm{pr}\{x\}$ and $\mathrm{sh}E = \bigcup_{x \in E}[x, \mathrm{pr}x]$. For a set E containing 0, we define $\mathrm{pr}E = S^d$.

Let $E_\rho(W) \subseteq S^d$ be defined by

$$E_\rho(W) = \mathrm{pr}\{WT(B_{2\rho}) \cap \mathrm{sh}\{W(B_{7\rho})\}\}$$

for fixed $T \in \Gamma$ and $W \in \Gamma$, $\rho > 0$. For some enumeration $W_1, W_2, \ldots$ of Γ, set

$$\mathcal{E}_\rho(T) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_\rho(W_k).$$

Making use of the results from [H], one can prove the following theorem.

THEOREM 4.1. *Let Γ be a nonelementary discrete group acting in B^{d+1}. If Γ is of divergence type then the conical limit set Λ_c has positive Patterson-Sullivan measure.*

As an application, we have the following result.

THEOREM 4.2. *For nonelementary discrete group Γ, there exists ρ_Γ such that if there exists $T \in \Gamma$ with $\mu_0[\mathcal{E}_{\rho_\Gamma}(T)] = 0$, then Γ is of convergence type.*

2. Patterson-Sullivan measure

For each $x \in B^{d+1}$, Patterson-Sullivan construct a measure μ_x by looking from x at the orbit under Γ of some point y. A unit object at the point γy appears from x to have size $e^{-(x,\gamma y)}$. Thus in dimension α we want to associate the scale factor $e^{-\alpha(x,\gamma y)}$ to the point γy. Now here is how μ_x is constucted. For s a positive real number, consider the infinite series $g_s(x,y) = \sum_{\gamma\in\Gamma} e^{-s(x,\gamma y)}$. For x and y fixed this series is proportional to

$$(1.1) \qquad \sum_{k=0}^{\infty} s_k e^{-ks}$$

where s_k is the number of points in a half-open annulus of radii in $(k-\frac{1}{2}, k+\frac{1}{2}]$ centered about x. The series (1.1) converges for $s > \delta$ and diverges for $s < \delta$ where $\delta = \lim_{k\to\infty} \frac{1}{k} \log s_k$. Since Γ is discrete, $s_k \leq ce^{dk}$ for some constant c depending on the minimal separation of the orbit points Γy. Thus $\delta \leq d$. If we define n_k to be the number of orbit points in the closed ball of radius $k + \frac{1}{2}$ about x, then $n_k = \sum_{i=0}^{k} s_i$, so we may also write $\delta = \lim_{k\to\infty} \frac{1}{k} \log n_k$. Using the triangle inequalities $(x,\gamma y) \leq (x,y) + (y,\gamma y)$ and $(x,\gamma y) \leq (y,\gamma y) - (x,y)$ yields

$$(1.2) \qquad e^{-s(x,y)} g_s(y,y) \leq g_s(x,y) \leq e^{s(x,y)} g_s(y,y).$$

In particular δ depends not on x but only on the discrete group Γ. Since we are interested in groups of divergence type at the critical exponent δ, we will assume that $\lim_{s\to\delta} \sum_\Gamma e^{-s(x,\gamma y)} = \infty$ for $s > \delta$, and this is true for all x, y using (1.2). Now we consider the family of measures $\mu_s(x) = \frac{1}{g_s(y,y)} \sum_\Gamma e^{-s(x,\gamma y)} D(\gamma y)$ where $D(\gamma y)$ is the unit Dirac mass at γy. Appealing to Helly's theorem, there is a measure in the limit as $s_i \to \delta$, namely, $\mu_x = \lim_{s_i\to\delta} \mu_{s_i}(x)$. Since $g_s(y,y) \to \infty$ as $s \to \delta$, μ_x is concentrated on the cluster points of the orbit $\Gamma(y)$. Thus μ_x is a measure on the limit set $\Lambda(\Gamma)$. For $x \in B^{d+1}$ let M_x be the collection of positive finite measures on $B^{d+1} \cup S^d$ with the base point x. Any measure μ_x belong to M_x satisfies

(a) μ_x is supported on the limit set $\Lambda(\Gamma)$.

(b) For any $x, z \in B^{d+1}$, μ_x and μ_z are absolutely continuous with respect to each other and the Radon-Nikodym derivative is $\{d\mu_x/d\mu_z\}(\zeta) = \{P(x,\zeta)/P(z,\zeta)\}^d$, where $\zeta \in S^d$ and $P(x,\zeta)$ is the Poisson kernel on B^{d+1}.

(c) For any Borel set E of $B^{d+1} \cup S^d$ and any $\gamma \in \Gamma$, we have $\mu_x(\gamma^{-1}(E)) = \mu_{\gamma(x)}(E)$.

3. Projections and conical limit points

In this section we will characterize the conical limit points in terms of projections and also state one result on groups of divergence type which will be useful later. First of all, we denote by $b(0 : a, \rho)$ the projection of the hyperbolic ball of center a and radius ρ from the origin onto S^d. We need the following result which can be found in Theorem 1.2.1 of [**N**] for the charaterization.

PROPOSITION 3.1. *Suppose $a \in B^{d+1}$ and $\xi, \eta \in S^d, \xi \neq \eta$. Let s be the hyperbolic distance from a to the geodesic joining ξ and η then*

$$\cosh s = \frac{2|a-\xi||a-\eta|}{|\xi-\eta|(1-|a|^2)}.$$

Note that a point $\xi \in S^d$ belongs to $b(0 : a, \rho)$ if and only if the radius to ξ passes within a hyperbolic distance ρ of a. By the above proposition, this is equivalent to

$$|a-\xi||a+\xi| < (1-|a|^2)\cosh\rho \tag{3.1}$$

With this result in hand, we next state the connection between conical limit points and projections.

PROPOSITION 3.2. *Let Γ be a discrete group acting in B^{d+1} and ξ is a conical limit point for Γ if and only if for some $a \in B^{d+1}$ and $\rho > 0$, ξ belongs to infinitely many projections $b(0 : \gamma(a), \rho) : \gamma \in \Gamma$.*

Proof. Suppose $\{a_n\}$ is a sequence of points in B^{d+1} such that $|a_n| \to 1$ as $n \to \infty$ and $\rho > 0$ is chosen. From (3.1), it follows that $\xi \in S^d$ belongs to the projections $b(0 : a_n, \rho)$, $n = 1, 2, \ldots$ if and only if there is a constant $k > 0$ such that for n large enough $|\xi - a_n| < k(1 - |a_n|)$. This implies, by Theorem 1 (vi) in [**B – M**], ξ is a conical limit point. Therefore this completes the proof of Proposition 3.2.

Using Agard's approach [**A**] together with properies of the Patterson-Sullivan measure μ_0, one can prove the following Theorem.

THEOREM 3.3. *For every nonelementary discrete group* Γ, *there exists* ρ_Γ *such that if there exists* $T \in \Gamma$ *with*

(1) $T(B_{\rho_\Gamma}) \cap B_{\rho_\Gamma} = \emptyset$ *and*
(2) $\mu_0[\mathcal{E}_{\rho_\Gamma}(T)] = 0$,

then Γ *is of convergence type.*

A detailed proof of the above theorem can be found in [**H**].

The following corollary is an immediate consequence of the Theorem 3.3.

COROLLARY 3.4. *If* Γ *is of divergence type, then* $\mu_0[\mathcal{E}_{\rho\Gamma}(T)] > 0$ *for all but finitely many* $T \in \Gamma$.

4. Conical limit sets and groups of divergence type

THEOREM 4.1. *Let* Γ *be a nonelementary discrete group acting in* B^{d+1}. *If* Γ *is of divergence type then the conical limit set* Λ_c *has positive Patterson-Sullivan measure.*

Proof. Suppose for the contradiction that $\mu_0(\Lambda_c) = 0$. If $p \in E_{\rho_\Gamma}(W_k)$ for infinitely many k. Hence $p \in \mathrm{pr}\{W_k T(B_{2\rho_\Gamma})\}$ for infinitely many k. By Proposition 3.2, $p \in \Lambda_c$. This shows $\mathcal{E}_{\rho_\Gamma}(T) \subseteq \Lambda_c$ for every $T \in \Gamma$. Since we assume that $\mu_0(\Lambda_c) = 0$, we have $\mu_0[\mathcal{E}_{\rho_\Gamma}(T)] = 0$ for every $T \in \Gamma$. Then by Theorem 3.3, Γ is of convergence type. This is a contradiction.

THEOREM 4.2. *For every nonelementary discrete group* Γ, *there exists* ρ_Γ *such that if there exists* $T \in \Gamma$ *with* $\mu_0[\mathcal{E}_{\rho_\Gamma}(T)] = 0$ *then* Γ *is of convergence type.*

Proof. If $T(B_{\rho_\Gamma}) \cap B_{\rho_\Gamma} = \emptyset$ then we are in the case of Theorem 3.3. Therefore we assume that $T(B_{\rho_\Gamma}) \cap B_{\rho_\Gamma} \neq \emptyset$. A point $\xi \in \mathcal{E}_{\rho_\Gamma}(T)$ if and only if $\xi \in E_{\rho_\Gamma}(W_k)$ for infinitely many W_k. Since $T(B_{\rho_\Gamma}) \cap B_{\rho_\Gamma} \neq \emptyset$, we have $W_k T(B_{2\rho_\Gamma}) \subset W_k(B_{7\rho_\Gamma})$ for all W_k. Therefore $\xi \in \mathcal{E}_{\rho_\Gamma}$ if and only if $\xi \in \mathrm{pr}\{W_k T(B_{2\rho_\Gamma})\}$ for infinitely many W_k if and only if there exists $k(2\rho_\Gamma) > 1$ such that

$$\frac{|\xi - W_k T(0)|}{1 - |W_k T(0)|} \leq k(2\rho_\Gamma).$$

Next we define, for $k \geq 1$, a point $\xi \in \Lambda_c(k)$ if there exists a sequence $\{\gamma_n\}$ such that

$$\lim_{n\to\infty} \frac{|\xi - \gamma_n(0)|}{1 - |W_k T(0)|} \leq k.$$

Then $\Lambda_c(k)$ is Γ-invariant and $\Lambda_c = \bigcup_{k=1}^{\infty} \Lambda_c(k)$. To show this, let $\xi \in \Lambda_c(k)$. Then there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\lim_{n\to\infty} \frac{|\xi - \gamma_n(0)|}{1 - |\gamma_n(0)|} \leq k.$$

Applying $g \in \Gamma$, since g is conformal and preserves S^d, we have

$$\lim_{n\to\infty} \frac{|g(\xi) - g\gamma_n(0)|}{1 - |g\gamma_n(0)|} \leq k.$$

Therefore $g(\xi) \in \Lambda_c(k)$. The second part is obvious. Suppose for the contradiction that Γ is of divergence type then $\mu_0(\Lambda_c) > 0$ by Theorem 4.1. Using Theorem 4.4.4 in [N], for each $k \geq 1, \Lambda_c(k)$ has either 0 or full measure. Hence there exists $n_0 \geq 1$ such that $\mu_0[\Lambda_c(n_0)] = \mu_0(\Lambda_c)$.

For $\rho_\Gamma > 0$ such that $k(2\rho_\Gamma) > n_0$, we have $\mathcal{E}_{\rho_\Gamma}(T) \supset \Lambda_c(n_0)$. Therefore $\mu_0[\mathcal{E}_{\rho_\Gamma}(T)] > 0$. This is a contradiction.

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