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EQUICONTINUITY OF ITERATES OF A MAP ON THE CIRCLE

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1. Introduction

Let f be a continuous function which maps the circle S^1 into itself and let $f^{n+1} = f \circ f^n$, for $n = 1, 2, \cdots$. Research in the social, biological and physical sciences often leads to a study of the sequence $\{f^n\}$ of iterates [4], [6]. Of importance to the researcher is the question, "if $d(x, y)$ is small, will $d(f^n(x), f^n(y))$ be small for all n? ". Then, with this function f, one can be almost sure that both the initial value x_0 and the estimate y_0 will be in S^1, and using $f^n(y_0)$ to predict $f^n(x_0)$ is of value. However, practical problems do not usually lead to the equicontinuity of $\{f^n\}$. We often find chaotic behavior of various sort. In mathematical language, one would be the family $\{f^n\}$ to be equicontinuous. In 1990, the conditions under which equicontinuity of the family of iterates $\{f^n\}$ of a continuous function f that maps a compact interval I into itself does occur were determined by Bruckner and Hu [1]. In fact, they showed that this happens only under exceptional circumstances.

The purpose of this paper is to determine conditions under which equicontinuity of the family of iterates $\{f^n\}$ of a continuous function that maps the circle S^1 into itself does occur. We shall see that equicontinuity of the family of iterates $\{f^n\}$ occurs only under special cases. Actually, we will show that this happens only for rotations when degree of the function is 1, and for involutions when degree of the function is -1.

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2. Main Results

Let $\pi : R \to S^1$ be the covering map. Define a metric d on the circle S^1 by $d(\pi(x), \pi(y)) = |x - y|$, where $x, y \in R$, $|x - y| < \frac{1}{2}$. Then d is a well-defined metric on S^1 which is equivalent to the original one. For the conveniency, we will use this metric d on S^1.

Let f be a continuous map from the circle S^1 into itself. Then the family of iterates $\{f^n\}$ of f is said to be *equicontinuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in S^1$, $|x - y| < \delta$ implies $|f^n(x) - f^n(y)| < \epsilon$ for all $n \in N$.

PROPOSITION 1. *Let $f : S^1 \to S^1$ be a continuous map, and F be a lifting of f. Then $\{f^n\}$ is equicontinuous if and only if so is $\{F^n\}$.*

Proof. Suppose that $\{F^n\}$ is equicontinuous and $\epsilon > 0$ is given. Then there exists $\delta > 0$ with $\delta < \frac{1}{2}$ such that if $|x-y| < \delta$, $|F^n(x)-F^n(y)| < \epsilon$ for all $n \in N$. Then for any $u, v \in S^1$ with $d(u, v) < \delta$, there exist $x, y \in R$ such that $\pi(x) = u, \pi(y) = v$ and $d(u, v) = |x - y| < \delta$. Then for all $n \in N$,

$$\begin{aligned} d\big(f^n(u), f^n(v)\big) &= d\big(f^n\big(\pi(x)\big), f^n\big(\pi(y)\big)\big) \\ &= d\big(\pi\big(F^n(x)\big), \pi\big(F^n(y)\big)\big) \\ &\leq |F^n(x) - F^n(y)| < \epsilon \end{aligned}$$

Therefore $\{f^n\}$ is equicontinuous.

For the converse, suppose that $\{f^n\}$ is equicontinuous and $\epsilon > 0$ is given with $\epsilon < \frac{1}{3}$. Then there exists $\delta > 0$ with $\delta < \frac{1}{3}$ such that if $d(u, v) < \delta$ for $u, v \in S^1$, $d\big(f^n(u), f^n(v)\big) < \epsilon$ for all $n \in N$. Now we claim that if $|x - y| < \delta$ for $x, y \in R$, then $|F^n(x) - F^n(y)| < \epsilon$ for all $n \in N$. On the contrary, we assume that there exist $x, y \in R$ and $n \in N$ such that $|x - y| < \delta$ and $|F^n(x) - F^n(y)| \geq \epsilon$. We know that $d\big(\pi(x), \pi(y)\big) = |x - y| < \delta$ and hence

$$d\big(f^n\big(\pi(x)\big), f^n\big(\pi(y)\big)\big) = d\big(\pi\big(F^n(x)\big), \pi\big(F^n(y)\big)\big) < \epsilon.$$

Then, in fact, $|F^n(x) - F^n(y)| \geq \frac{2}{3}$. To show this fact, we know that there exsits $z \in \pi^{-1}(\pi(F^n(y))$ such that

$$|F^n(x) - z| = d\big(\pi\big(F^n(x)\big), \pi\big(F^n(y)\big)\big) < \epsilon, \quad \text{and} \quad |z - F^n(y)| \geq 1.$$

Let $x_1 = x, y_1 = y$ and $x_2 = (x_1 + y_1)/2$, then one of $|F^n(x_1) - F^n(x_2)|$ and $|F^n(y_1) - F^n(x_2)|$ is greater than or equal to $\frac{1}{3}$. In particular, since $|x_1 - x_2| < \frac{\delta}{2}$ and $|y_1 - x_2| < \frac{\delta}{2}$, by the same way, one of $|F^n(x_1) - F^n(x_2)|$ and $|F^n(y_1) - F^n(x_2)|$ is greater than or equal to $\frac{2}{3}$. Put one of these x_1 and y_1 by y_2. By induction argument we can find x_m, y_m such that $x_m = (x_{m-1} + y_{m-1})/2$ and $|x_m - y_m| < 2^{-m+1}\delta$ and $|F^n(x_m) - F^n(y_m)| \geq \frac{2}{3}$.

This contradicts the continuity of F^n and completes the proof.

The following Lemma 2 is due to [1, Corollary 8] and [7].

LEMMA 2. *Let (X, d) be a compact metric space and $f : X \to X$ be a surjective map whose sequence of iterates $\{f^n\}$ is equicontinuous. Then f is a homeomorphism.*

We say that two maps $f, g : X \to X$ are *topologically conjugate* if there exists a homeomorphism α of X such that $f = \alpha^{-1} g \alpha$.

LEMMA 3. *Let (X, d) be a compact metric space and $f, g : X \to X$ be topologically conjugate. Then $\{f^n\}$ is equicontinuous if and only if so is $\{g^n\}$.*

Proof. Let $\{g^n\}$ be equicontinuous and $\epsilon > 0$ be given. Then there exists $\delta' > 0$ such that $d(x, y) < \delta'$ implies $d(\alpha^{-1}(x), \alpha^{-1}(y)) < \epsilon$. Also there exists $\delta'' > 0$ such that $d(x, y) < \delta''$ means $d(g^n(x), g^n(y)) < \delta'$ for all $n \in N$. We can find $\delta > 0$ such that $d(\alpha(x), \alpha(y)) < \delta''$ whenever $d(x, y) < \delta$. Then for $x, y \in X$ with $d(x, y) < \delta$, $d(f^n(x), f^n(y)) = d(\alpha^{-1} g^n \alpha(x), \alpha^{-1} g^n \alpha(y)) < \epsilon$, for all $n \in N$.

Similary the converse is obvious.

LEMMA 4. *Let f be a map from the circle S^1 into itself such that f has no periodic points and $\{f^n\}$ is equicontinuous. Then the orbit $\{f^n(x)\}$ is dense in S^1 for all $x \in S^1$.*

Proof. First we note $\deg(f) = 1$, because f has no periodic points, and hence f has to be a homeomorphism by Lemma 2. Let $x \in S^1$ be given and $\omega(x)$ be the set of ω-limits of x under f, i.e., $\omega(x) = \{y \in S^1 | f^{n_i}(x) \to y$ for some $n_i \to \infty\}$. Then we know that $\omega(x)$ is closed and invariant under f. Now we claim $\omega(x) = S^1$, which asserts our conclusion. Suppose that there exists $z \in S^1$ with $z \notin \omega(x)$. Let I_z be the connected component of $S^1 \setminus \omega(x)$ containing z. Then I_z is an open arc containing z. By notations, for any $a, b \in S^1$, we denote the closed arc from a to b in the counterclockwise direction by $[a, b]$ and $(a, b) = [a, b] - \{a, b\}$. Then there are two points $a, b \in S^1$ such that $I_z = (a, b)$. Then note that $a \neq b$, since f has no periodic points and hence $w(x)$ is not a singleton. According to Proposition 7 of [1] we have an increasing sequence $\{n_i\}$ of positive integers such that $f^{n_i}(a) \to a$ and $f^{n_i}(b) \to b$. Since $a, b \in \omega(x)$ and $\omega(x) \cap I_z = \phi$, $f^{n_i}(a), f^{n_i}(b) \notin I_z$ for all i. We may assume that $f^{n_k}(a) \in (f^{n_k}(b), a)$ and $f^{n_k}(b) \in (b, f^{n_k}(a))$, that is , $[a, b] \subseteq [f^{n_k}(a), f^{n_k}(b)] = f^{n_k}([a, b])$ for some positive integer k. The last equation follows from that f is a homeomorphism of degree 1. This means that f^{n_k} has a fixed point in $[a, b]$, which is a contradiction.

THEOREM 5. *Let f be a continuous map of the circle. Then $\{f^n\}$ is equicontinuous if and only if one of the followings holds:*

(i) $\deg(f) = 0$ *and* $\cap_{n=1}^{\infty} f^n(S^1) = F_2$, *where* F_2 *is the fixed point set of* f^2;

(ii) $\deg(f) = 1$ *and* f *is topologically conjugate to a rotation map;*

(iii) $\deg(f) = -1$ *and* f^2 *is the identity.*

Proof. (i) Suppose that $\{f^n\}$ is equicontinuous. Then by Lemma 2, if f is surjective, then f must be a homeomorphism. Therefore we declare that $\deg(f) \in \{0, -1, 1\}$. Further, if $\deg(f) = 0$, then f is not surjective. In this case, we may assume that there exists a lifting F of f such that $F(R) \subset [0, 1]$. Now by putting $G = F|_{[0,1]}$, we know that $\{G^n\}$ is equicontinuous if and only if $\cap_{n=1}^{\infty} G^n(I)$ is the fixed point set of G^2 by Corollary 12 of [1]. Therefore $\pi(\cap_{n=1}^{\infty} G^n(I)) = \cap_{n=1}^{\infty} f^n(S^1)$ is the fixed point set of f^2. Conversely, if $\deg(f) = 0$ and $\cap_{n=1}^{\infty} f^n(S^1) = F_2$ is the fixed point set of f^2, then f is not surjective, because if f is surjective, then $F_2 = S^1$ and hence $\deg(f^2) = 1$. Hence by the same way in the above, we may assume that there exists a lifting F of f with $F(R) \subseteq [0, 1)$ and put $G = F|_{[0,1]}$. Then by putting $\tau = \pi|_{[0,1)}$, we have that

$$\tau^{-1}(\cap_{n=1}^{\infty} f^n(S^1)) = \cap_{n=1}^{\infty} G^n(I)$$

is the fixed point set of G^2. Therefore by Corollary 12 of [1], $\{G^n\}$ is equicontinuous, and hence by Proposition 1, $\{f^n\}$ has to be equicontinuous.

(ii) According to Lemma 3, we need merely to prove the only if part. Let $\deg(f) = 1$. Then, by Lemma 2., f is a homeomorphism and hence the rotational number $\rho(f)$ is uniquely determined up to integers. Firstly, suppose that $\rho(f)$ is irrational. Then it is well-known that f is topologically conjugate to the rotation map $\pi(x) \mapsto \pi(x + \rho(f))$ if $\{f^n(x)\}$ is dense in S^1 for some $x \in S^1$ (see [2],[3] and [5]). Hence by using Lemma 4, we obtain the desired result. Suppose that $\rho(f) = r/n$ with $(n, r) = 1$ is the rotation number. Then f has a periodic point of period n. Therefore we can take a lifting G of f^n such that G has a fixed point. Since also $\{G^m\}$ is equicontinuous, G is the identity, so that f^n is the identity. That is, all point in S^1 are periodic points of f of the same period n. Let $\{x_1, x_2, \cdots, x_n\}$ be an orbit of x_1 under f such that $x_1, x_2, \cdots, x_n$ lie on S^1 by the counterclockwise order. Let $y_i = \pi(\frac{i-1}{n})$ for $1 \le i \le n$. And let g be the rotation with the rotation number $\rho(g) = r/n$. Then $g(y_i) = y_j$ iff $f(x_i) = x_j$. We claim that f and g are topologically conjugate. By the above notation, the closed arc from x_i to x_{i+1} (or from y_i to y_{i+1}) in S^1 in the counterclockwise is denoted by $[x_i, x_{i+1}]$, (or $[y_i, y_{i+1}]$) where $1 \le i \le n$ $(n \ge 2)$, for $i = n$ we mean $x_{n+1} = x_1$ (or $y_{n+1} = y_1$). Define $\alpha : [y_1, y_2] \to [x_1, x_2]$ such that $\alpha(y_1) = x_1$, $\alpha(y_2) = x_2$ and α is a homeomorphism. Let $f(x_1) = x_i$ for some $1 \le i \le n$. Then $f(x_2) = x_{i+1}$, $g(y_1) = y_i$ and $g(y_2) = y_{i+1}$. Define $\alpha : [y_i, y_{i+1}] \to [x_i, x_{i+1}]$ by $\alpha(y) = f\alpha g^{-1}(y)$. Then $\alpha|_{[y_i, y_{i+1}]}$ is a homeomorphism. Now let $f(x_j) = x_1$ and $f(x_{j+1}) = x_2$ for some j. Then also $g(y_j) = y_1$ and $g(y_{j+1}) = y_2$. With this processing, we can define finally $\alpha : [y_j, y_{j+1}] \to [x_j, x_{j+1}]$ which is a homeomorphism. Then $\alpha : S^1 \to S^1$ is a homeomorphism, we need to prove that for $y \in [y_1, y_2]$, $\alpha(y) = f\alpha g^{-1}(y)$. But we have for $y \in [y_1, y_2]$,

$$\begin{aligned} f\alpha g^{-1}(y) &= f(f\alpha g^{-1})g^{-1}(y) \\ &= f^n \alpha (g^{-1})^n (y) \\ &= \alpha(y), \end{aligned}$$

since f^n and g^n are identity. That is, we have a homeomorphism α of S^1 such that $\alpha = f\alpha g^{-1}$. Since $g = \alpha^{-1}\alpha g = \alpha^{-1} f\alpha g^{-1} g = \alpha^{-1} f\alpha$, f and g are topologically conjugate.

(iii) Let $\deg(f) = -1$ and F be a lifting of f. Then F must have a fixed point. And then $\deg(f^2) = 1$ and F^2 has a fixed point. Since $\{F^n\}$ is equicontinuous, the fixed point set of F^2 is connected. Therefore F^2 is the identity, and hence f^2 is the identity.

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