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ON THE MONOTONICITY OF THE DITTERT FUNCTION ON CLASSES OF NONNEGATIVE MATRICES

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1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, and let J_n denote the $n \times n$ matrix all of whose entries are $1/n$. The *permanent* of a real $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\text{per } A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \tag{1.1}$$

where σ runs over all permutations of $\{1, \cdots, n\}$.

For $k \in \{1, \cdots, n\}$, let $\sigma_k(A)$ denote the sum of all subpermanents of order k of A.

The famed van der Waerden - Egoryĉev - Falikman theorem [3],[4] asserts that the permanent function attains its minimum over Ω_n uniquely at J_n.

In [4], Friedland and Minc remarked that a stronger version of this theorem is the following.

MONOTONICITY CONJECTURE. The permanent function is monotone decreasing on the line segment from $A \in \Omega_n$ to J_n.

It is often referred to as the *monotonicity of permanent* (abb. MP). MP has been proved for several classes of matrices in Ω_n [5],[6],[7],[9],[10],[13],[15].

A conjecture related to MP is the following one proposed by Doković [2] :

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DOKOVIĆ CONJECTURE. Let $A \in \Omega_n$. Then

$$\sigma_k(A) \geq \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A), \qquad k = 2, \cdots, n \tag{1.2}$$

with equality holds if and only if $A = J_n$.

The Doković conjecture for $k \leq 3$ was proved by Doković himself [2].

In [11], Massoud Malek-Shahmirzadi has revealed a connection between the monotonicity conjecture and the Doković conjecture by showing that

THEOREM A. *If $A \in \Omega_n$ satisfies the Doković inequality* (1.2) *then MP holds for A.*

For a positive integer n, let K_n denote the set of all real nonnegative $n \times n$ matrices whose entries have sum n. For $X \in K_n$ with row sums $r_1, \cdots, r_n$ and column sums $c_1, \cdots, c_n$, let

$$\varphi(X) = \prod_{i=1}^{n} r_i + \prod_{j=1}^{n} c_j - \text{per } X. \tag{1.3}$$

Then φ defines a real valued function on K_n. We shall call φ the *Dittert function.* The following conjecture [12] due to E. Dittert is still open.

DITTERT CONJECTURE. The Dittert function attains its maximum over K_n uniquely at J_n.

Clearly the Dittert conjecture is a generalization of van der Waerden - Egoryĉev - Falikman theorem.

For $k \in \{1, \cdots, n\}$, let $Q_{k,n}$ denote the set of all strictly increasing integer sequences of length k chosen from $1, \cdots, n$. For $\alpha, \beta \in Q_{k,n}$, and for an $n \times n$ matrix A, let $A[\alpha|\beta]$ denote the $k \times k$ submatrix of A lying in rows α and columns β, and $A(\alpha|\beta)$ is the complement of $A[\alpha|\beta]$ in A.

Let φ_k denote the real valued function defined on K_n by

$$\varphi_k(A) = \sum_{\alpha,\beta\in Q_{k,n}} \left(\prod_{i\in\alpha} r_i + \prod_{j\in\beta} c_j - \operatorname{per} A[\alpha|\beta] \right) \tag{1.4}$$

where r_i and c_j are the sum of all the entries in row i and the sum of all the entries in column j of $A \in K_n$, respectively. Note that $\varphi_n = \varphi$. We call φ_k the k-th *sub-Dittert function.*

In this paper, we study the *monotonicity of the Dittert function* (abb. MD) on the line segment from $A \in K_n$ to J_n generalizing both the Dittert conjecture and the Monotonicity conjecture for permanent, and obtain a sufficient condition on $A \in K_n$ for which the MD holds. It is also proved that if $A \in K_n$ satisfies the Doković inequality (1.2) then MD holds for A, and a subclass of K_n for which MD holds is found.

2. Some Preliminary Lemmas

For $k \in \{1, \cdots, n\}$, let S_k denote the k-th elementary symmetric function of $\mathbf{R}^n$, *i.e.*,

$$S_k(\mathbf{x}) = \sum_{\alpha\in Q_{k,n}} \prod_{i\in\alpha} x_i \tag{2.1}$$

for $\mathbf{x} = (x_1, \cdots, x_n)^T \in \mathbf{R}^n$.

To study the monotonicity of the Dittert function, we need the following results concerning elementary symmetric functions and sub-Dittert functions. The following lemmas are due to Cheon and Hwang [1].

LEMMA 2.1. [1] *Let $A \in K_n$. Then*

$$\varphi_2(A) \le \varphi_2(J_n). \tag{2.2}$$

LEMMA 2.2. [1] *Let* $\mathbf{x} = (x_1, \cdots, x_n)^T$ *be a nonnegative real vector such that* $x_1 + \cdots + x_n = n$ *and let* $k \in \{1, \cdots, n\}$. *Then*

$$S_k(\mathbf{x}) \leq \binom{n}{k}. \tag{2.3}$$

LEMMA 2.3. [14] *For* $k \in \{1, \cdots, n\}$, *the function* S_k/S_{k-1} *is Schur-concave on the set of all positive real vectors.*

COROLLARY 2.1. *Let* $\mathbf{x} = (x_1, \cdots, x_n)^T$ *be a positive real vector such that* $x_1 + \cdots + x_n = n$ *and let* $k \in \{1, \cdots, n\}$. *Then*

$$\frac{S_k(\mathbf{x})}{S_{k-1}(\mathbf{x})} \leq \frac{n-k+1}{k}. \tag{2.4}$$

Proof. The inequality (2.4) follows directly from Lemma 2.3 and Lemma 2.2.

In addition to these lemmas, we make use of the following theorem due to Massoud Malek-Shahmirzadi.

LEMMA 2.4. [11] *Let* A *be an* $n \times n$ *matrix and* x *a real number. Then*

$$\operatorname{per}(A + nxJ_n) = \sum_{k=0}^{n} (n-k)! \sigma_k(A) x^{n-k}. \tag{2.5}$$

3. Monotonicity of The Dittert Function

Let $A \in K_n$ have row sum vector R and column sum vector C. Then for each $k = 1, \cdots, n$, $\varphi_k(A)$ can be written as

$$\varphi_k(A) = \binom{n}{k} S_k(R) + \binom{n}{k} S_k(C) - \sigma_k(A). \tag{3.1}$$

For an $n \times n$ matrix A and for $\delta_k := n! n^k / k! n^n$, let

$$\lambda_k(A) := \varphi_k(A) + (1 - \delta_k)\sigma_k(A). \tag{3.2}$$

We are now ready to prove one of our main theorems.

THEOREM 3.1. *Let $A \in K_n$ satisfy the condition*

$$\lambda_k(A) \leq \left(\frac{n-k+1}{k}\right)^2 \lambda_{k-1}(A), \qquad k = 1, \cdots, n. \tag{3.3}$$

Then MD holds for A.

Proof. Let A be a matrix on K_n with row sum vector $R = (r_1, \cdots, r_n)^T$ and column sum vector $C = (c_1, \cdots, c_n)^T$. For real θ, let $A_\theta = (1-\theta)A + \theta J_n$ and $r_i(\theta) = r_i + (1-r_i)\theta$, $c_j(\theta) = c_j + (1-c_j)\theta$ for $i, j = 1, \cdots, n$ and let $R_\theta = (r_1(\theta), \cdots, r_n(\theta))^T$ and $C_\theta = (c_1(\theta), \cdots, c_n(\theta))^T$. Then

$$\varphi(A_\theta) = \prod_{i=1}^{n} r_i(\theta) + \prod_{j=1}^{n} c_j(\theta) - \text{per } A_\theta. \tag{3.4}$$

We prove that if $A \in K_n$ satisfies the condition (3.3) then $\varphi'(A_\theta) \geq 0$ for the interval $0 < \theta < 1$.

Let $nx := \frac{\theta}{1-\theta}$ $(0 < \theta < 1)$. Then

$$A_\theta = \frac{1}{1+nx}(A + nxJ_n),$$

$$r_i(\theta) = \frac{1}{1+nx}(r_i + nx) \quad and \quad c_j(\theta) = \frac{1}{1+nx}(c_j + nx).$$

We define

$$g(x) := \prod_{i=1}^{n} r_i(\theta) = \frac{1}{(1+nx)^n} \prod_{i=1}^{n}(r_i + nx),$$

$$h(x) := \prod_{j=1}^{n} c_j(\theta) = \frac{1}{(1+nx)^n} \prod_{j=1}^{n}(c_j + nx),$$

$$p(x) := \text{per } A_\theta = \frac{1}{(1+nx)^n}\text{per } (A + nxJ_n),$$

and

$$f(x) := \varphi(A_\theta) = g(x) + h(x) - p(x)$$

on the interval $0 < \theta < 1$. Then we get

$$g'(x) = \frac{-n^2}{(1+nx)^{n+1}}\prod_{i=1}^{n}(r_i+nx) + \frac{1}{(1+nx)^n}\prod_{i=1}^{n}(r_i+nx)\sum_{i=1}^{n}\frac{n}{r_i+nx}. \tag{3.5}$$

We compute first that

$$\prod_{i=1}^{n}(r_i+nx) = \sum_{k=0}^{n} S_k(R)(nx)^{n-k} \tag{3.6}$$

and

$$\prod_{i=1}^{n}(r_i+nx)\sum_{i=1}^{n}\frac{1}{r_i+nx} = \sum_{k=1}^{n}(n-k+1)S_{k-1}(R)(nx)^{n-k}. \tag{3.7}$$

From (3.5), (3.6) and (3.7) it follows that

$$g'(x) = \frac{1}{(1+nx)^{n+1}}\sum_{k=1}^{n}\Big\{(n-k+1)S_{k-1}(R) - kS_k(R)\Big\}n^{n-k+1}x^{n-k}. \tag{3.8}$$

Similarly we can show that

$$h'(x) = \frac{1}{(1+nx)^{n+1}}\sum_{k=1}^{n}\Big\{(n-k+1)S_{k-1}(C) - kS_k(C)\Big\}n^{n-k+1}x^{n-k}. \tag{3.9}$$

On the other hand, from (2.5), we get

$$p'(x) = \frac{1}{(1+nx)^{n+1}}\sum_{k=1}^{n}(n-k)!\{(n-k+1)^2\sigma_{k-1}(A) - nk\sigma_k(A)\}x^{n-k}. \tag{3.10}$$

Thus from (3.8), (3.9) and (3.10), we get

$$f'(x) = \frac{1}{(1+nx)^{n+1}}\sum_{k=1}^{n}\widehat{T}_k(A)x^{n-k} \tag{3.11}$$

where

$$\begin{aligned}\hat{T}_k(A) &\\ = n^{n-k+1}&\{(n-k+1)(S_{k-1}(R)+S_{k-1}(C))-k(S_k(R)+S_k(C))\}\\ +(n-k)!&\left\{nk\sigma_k(A)-(n-k+1)^2\sigma_{k-1}(A)\right\}. \qquad (3.12)\end{aligned}$$

From (3.1), we have

$$S_k(R)+S_k(C)=(\varphi_k(A)+\sigma_k(A))/\binom{n}{k}. \qquad (3.13)$$

By an elementary computation, we get, from (3.2) and (3.13)

$$\widehat{T}(A)=\frac{kn^{n-k+1}}{\binom{n}{k}}\left\{\left(\frac{n-k+1}{k}\right)^2\lambda_{k-1}(A)-\lambda_k(A)\right\}. \qquad (3.14)$$

Hence from (3.11) and (3.14), if

$$\lambda_k(A)\le\left(\frac{n-k+1}{k}\right)^2\lambda_{k-1}(A)$$

then $f'(x)\ge 0$, which completes the proof.

In the Theorem 3.1, if A is restricted to be in Ω_n then the condition (3.3) coincides with the Doković inequality (1 2).

Note that if $k=n$ in (3.3) then

$$\varphi(A)\ \le\ \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ij}(A) \qquad (3.15)$$

where

$$\varphi_{ij}(A)=\prod_{k\ne i}r_k+\prod_{k\ne j}c_k-\operatorname{per}A(i|j).$$

In [8], it was shown that not every matrix $A \in K_n$ satisfies the condition (3.15). But we guess that MD holds for every matrix in K_n satisfying (3.15).

For an $n \times n$ matrix A, and for each $k = 1, \cdots, n$, let

$$T_k(A) := \lambda_k(A) - \left(\frac{n-k+1}{k}\right)^2 \lambda_{k-1}(A). \tag{3.16}$$

A simple computation shows that $T_k(J_n) = 0$ for each $k = 1, \cdots, n$. The Theorem 3.1 just says that if $T_k(A) \leq 0$ for $k = 1 \cdots, n$, then MD holds for $A \in K_n$.

THEOREM 3.2. *The condition* (3.3) *holds for* $k \leq 2$.

Proof. The case $k = 1$ is trivial. To prove the theorem for $k = 2$, let A be a matrix on K_n with row sum vector R and column sum vector C. Then

$$\begin{aligned} T_2(A) &= \lambda_2(A) - \left(\frac{n-1}{2}\right)^2 \lambda_1(A) \\ &= \varphi_2(A) + (1-\delta_2)\sigma_2(A) - \binom{n}{2}^2\left(2 - \frac{n!}{n^n}\right). \end{aligned} \tag{3.17}$$

By an elementary computation, we get, from (3.1) and (3.17)

$$T_2(A) = \delta_2(\varphi_2(A) - \varphi_2(J_n)) + \binom{n}{2}(1-\delta_2)\left(S_2(R) + S_2(C) - 2\binom{n}{2}\right).$$

Thus from (2.2) and (2.3), it follows that

$$T_2(A) \leq 0,$$

which completes the proof.

As a corollary to Theorem 3.2, it follows that if $A \in K_n$ satisfies the condition (3.15) then MD holds for $n \leq 3$.

In the following, we find a subclass of K_n for which MD holds.

THEOREM 3.3. *For an $n \times n$ diagonal matrix D and for any $n \times n$ permutation matrices P and Q, if A is a matrix of the form $A = PDQ \in K_n$, then MD holds for A.*

Proof. First, let A be a positive diagonal matrix on K_n with row sum vector R and column sum vector C. Note $S_k(R) = S_k(C) = \sigma_k(A)$ for each $k = 1, \cdots, n$. Then

$$\lambda_k(A) = S_k(R)\left(2\binom{n}{k} - \delta_k\right). \tag{3.18}$$

Thus we get

$$\begin{aligned} T_k(A) &= \lambda_k(A) - \left(\frac{n-k+1}{k}\right)^2 \lambda_{k-1}(A) \\ &= S_{k-1}(R)\left\{\left(\frac{S_k(R)}{S_{k-1}(R)} - \frac{n-k+1}{k}\right)\left(2\binom{n}{k} - \delta_k\right) \right. \\ &\quad \left. - \left(\frac{n-k+1}{nk}\right)(k-1)\delta_k\right\} \end{aligned}$$

Hence from (2.4), it follows that

$$T_k(A) \leq 0$$

for $k = 1, \cdots, n$.

Now let A be a nonnegative diagonal matrix on K_n with $A := diag(a_1, \cdots, a_i, 0, \cdots, 0)$, and for a sufficiently small $\epsilon > 0$, let $A(\epsilon) := diag(a_1 - \epsilon, \cdots, a_i - \epsilon, \epsilon', \cdots, \epsilon')$ where $\epsilon' = i\epsilon/(n-i)$. Then $A(\epsilon)$ is a positive diagonal matrix on K_n and it follows that for each $k = 1, \cdots, n$,

$$T_k(A) = \lim_{\epsilon \to 0} T_k(A(\epsilon)) \leq 0.$$

It shows the conclusion also holds for nonnegative matrices, hence the proof is completed.

Theorem 3.3 has the following corollary as a special case.

COROLLARY 3.1. *MP holds for any permutation matrix.*

THEOREM 3.4. *If $A \in K_n$ satisfies the Doković inequality* (1.2) *for each $k = 1, \cdots, n$, then MD holds for A.*

Proof. Let $A \in K_n$. What we proved in the proof of the Theorem 3.3 enables us to assume that both the row sum vector R and the column sum vector C of A are positive. Note that

$$\lambda_k(A) = \binom{n}{k}\Big(S_k(R) + S_k(C)\Big) - \delta_k \sigma_k(A). \tag{3.19}$$

We get from (3.19)

$$\begin{aligned} T_k(A) &= \lambda_k(A) - \left(\frac{n-k+1}{k}\right)^2 \lambda_{k-1}(A) \\ &= S_{k-1}(R)\binom{n}{k}\left(\frac{S_k(R)}{S_{k-1}(R)} - \frac{n-k+1}{k}\right) \\ &\quad + S_{k-1}(C)\binom{n}{k}\left(\frac{S_k(C)}{S_{k-1}(C)} - \frac{n-k+1}{k}\right) \\ &\quad - \delta_k D_k(A), \end{aligned}$$

where

$$D_k(A) := \sigma_k(A) - \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A).$$

From (2.4), it follows that

$$T_k(A) \ \le\ -\delta_k D_k(A).$$

Thus if $\quad D_k(A) \ge 0 \quad$ for each $k = 1, \cdots, n$, then

$$T_k(A) \ \le\ 0,$$

which completes the proof.

Note that our Theorem 3.4 is a direct generalization of the Theorem A.

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