Bull. Korean Math. Soc. **30** (1993), No. 2, pp. 277–284

APPROXIMATE CONTROLLABILITY OF DELAY VOLTERRA CONTROL SYSTEM

JONG WON RYU†, JONG YEOUL PARK AND YOUNG CHEL KWUN

1. Introduction

We consider the abstract delay Volterra control system described by the following integral form:

$$
\begin{aligned}
x(t) &= x_t(\phi)(0) \\
&= U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi)) + (Bv)(s)\}ds \qquad (1)\\
x_0(\phi) &= \phi \in C
\end{aligned}
$$

Here, X and V are Hilbert space. The state function $x(t), 0 \le t \le T$, takes values in X; the control function v is given in $L^2(0, T : V)$; and $U(t,s)$ is a linear evolution operator on X. Let C be the Banach space of all continuous functions from an interval of the form $I = [-h, 0]$ to X with the norm defined by supremum. If a function u is continuous from $I \cup [0,T]$ to X, then u_t is an element in C, which has pointwise definition

$$u_t(\theta) = u(t+\theta), \text{ for } \theta \in I.$$

We assume that F is a nonlinear function from $[0,T] \times C$ to X.

The purpose of this paper is to prove the approximate controllability results for the delay Volterra system in the case of trajectories. In method, our paper differs from [5].

Received August 24, 1992.
† Supported by a Grant of Dong-A University Foundation,1991.

2. Preliminaries

In this section, the norm of the space $L^2(0,T:X)$ or $L^2(0,T:V)$ is denoted by $\|\cdot\|$, for the other space, we use $\|\cdot\|_X, \|\cdot\|_C$, and so on. We consider a unique mild solution, for each u in $L^2(0,T:V)$,

$$(2) \quad x_t(\phi:u)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s,x_s(\phi:u)) + u(s)\}ds.$$

In this case, we can define a solution mapping W from $L^2(0,T:V)$ to $C(0,T:C)$ by

$$(Wu)(t) = x_t(\phi:u)(\cdot).$$

For convenience, let us define a linear operator $\tilde{S}$ from $L^2(0,T:X)$ to $C(0,T:X)$ by

$$(\tilde{S}p)(t) = \int_0^t U(t,s)p(s)ds \quad p \in L^2(0,T:X), \ 0 \le t \le T.$$

The following assumptions will be made in the remaining of the paper:

(H1) For all $p(\cdot) \in L^2(0,T:X)$, there exists $q(\cdot) \in \overline{R(B)}$ ($R(B)$ is the range of B) such that

$$\tilde{S}p = \tilde{S}q.$$

(H2) The solution mapping $W : L^2(0,T:V) \to C(0,T:C)$ is compact (i.e., W is continuous and maps bounded sets into compact sets).

For each $t \in [0,T]$, define $N_t = \{p \in L^2(0,T:X) : \tilde{S}p(t) = 0\}$ and put $N = \cap_{0\le t\le T} N_t$. In $L^2(0,T:X)$, we denote the orthogonal subspace of N by $N^\perp$, the projection with the range $N^\perp$ by G, the range space of the operator B by $R(B)$, and its closure by $\overline{R(B)}$.

Define $P : N^\perp \to \overline{R(B)}$ as follow. For all $u \in N^\perp$, Pu is the unique minimum norm element in $\{u+N\} \cap \overline{R(B)}$. Thus

$$\|Pu\| = \min\{\|y\| : y \in \{u+N\} \cap \overline{R(B)}\}.$$

It can be shown that under hypothesis (H1), p is well defined, linear and continuous([4, Lemma 1]).

Next, we define another function $\mathcal{F} : L^2(0,T : C) \to L^2(0,T : X)$ by

$$(\mathcal{F}x)(t) = F(t, x_t(\cdot)).$$

Since $W : L^2(0,T : V) \to C(0,T : C)$ is compact, and $F : [0,T] \times C \to X$ is locally Lipschitz in $\psi \in C$, uniformly in $t \in [0,T]$, it can be shown that $\mathcal{F}(\cdot) : L^2(0,T : C) \to L^2(0,T : X)$ is compact. Let $\tilde{X}$ be the quotient space $L^2([0,T] : X)/N$. Then $\|\tilde{x}\| = \inf\{\|u + y\| : y \in N\}$ is a norm on $\tilde{X}$. Next, let G denote the isometric isomorphism from $\tilde{X}$ onto $N^{\perp}$, and $\tilde{\mathcal{F}} : \tilde{X} \to \tilde{X}$ be defined by $\tilde{\mathcal{F}}\tilde{u} = \mathcal{F}(PG\tilde{u}) + N$, for all $\tilde{u} \in \tilde{X}$. Again $\tilde{\mathcal{F}}$ is compact.

3. Approximate Controllability

First, we define the reachable set $K(F)$ in $C(0,T : X)$ by

$$K(F) = \{x_t(\phi : u)(0) \in C(0,T : X); x_t(\phi : u)(0) = U(t,0)\phi(0) + \int_0^t U(t,s)\{F(s, x_s(\phi : u)) + (Bu)(s)\}ds, \quad u \in L^2(0,T : U)\},$$

and

$$K(0) = \{z \in C(0,T : X); z(t) = U(t,0)\phi(0) + \int_0^t U(t,s)Bu(s)ds, \quad u \in L^2(0,T : U)\}.$$

THEOREM 1. *Under hypothesis (H1), we have*

$$\overline{K(F)} \subset \overline{K(0)}$$

Proof. First, we show that $K(F) \subset \overline{K(0)}$. Let $\eta = x_t(\phi : u)(0) \in K(F)$. Then, from hypothesis (H1), it follows that there exists a function $q \in \overline{R(B)}$ such that $\eta(t) = \tilde{S}q(t)$, for all $t \in [0,T]$. For every $\varepsilon > 0$, there exists a control function $u_\varepsilon \in U$ such that

$$\|q - Bu_\varepsilon\| \le (M\sqrt{T})^{-1}\varepsilon = \varepsilon_0.$$

Put $y_\varepsilon(t) = \tilde{S}Bu_\varepsilon(t)$. Then, we have

$$\|\eta(t) - y_\varepsilon(t)\| \le \int_0^t \|U(t,s)q(s) - U(t,s)Bu_\varepsilon(s)\|ds$$
$$\le M\sqrt{t}\varepsilon_0 \le \varepsilon, \quad 0 \le t \le T.$$

Since $y_\varepsilon \in K(0)$ on $[0,T]$, we see that $K(F) \subset \overline{K(0)}$. Consequently, we have $\overline{K(F)} \subset \overline{K(0)}$.

For the convenient the norm of $F : [0,T] \times C \to X$ denoted by $\|F(\cdot,\cdot)\|_X = \|F(\cdot,\cdot)\|$.

LEMMA 1. *Let $F : [0,T] \times C \to X$ be continuous in t, locally Lipschitz in $\psi \in C$, uniformly in $t \in [0,T]$, and $\lim_{\|\psi\|\to\infty} \|F(t,\psi)\|/\|\psi\| = c$ uniformly in $t \in [0,T]$. If $\sigma : R^+ \to R^+$ be a continuously differentiable, monotonically nondecreasing function such that $\|F(t,\psi)\| \le \sigma\|\psi\|$, $\forall(t,\psi) \in [0,T] \times C$, and $\lim_{r\to\infty} \sigma(r)/r = c$, then*

$$\|F(\cdot, Wu(\cdot))\| = \|F(\cdot, x_\cdot(\phi : u)(\cdot))\|$$
$$\le \sqrt{T}\sigma(M\phi(0) + M\sqrt{T}\|u\|)e^{MM_\sigma T},$$

where $M_\sigma = \sup\{\sigma'(r) : r \in R^+\} < \infty$ and $\sigma'(r)$ is derivative of $\sigma(r)$, $r \in R^+$.

proof. The system (2), first we prove that

$$\|x_t(\phi : u)(\theta)\|_X$$
$$= \|x_{t+\theta}(\phi : u)(0)\|_X$$
$$\le M\|\phi(0)\| + M \int_0^{t+\theta} \{\|F(s, x_s(\phi : u))\| + \|u(s)\|\}ds$$
$$\le M\|\phi(0)\| + M \int_0^{t+\theta} \|F(s, x_s(\phi : u))\|ds$$
$$+ M\|u\|\sqrt{t+\theta}, \qquad -h \le \theta \le 0.$$

Hence

$$\sup_{-h\le\theta\le 0} \|x_t(\phi : u)(\theta)\|_X$$
$$\le M\|\phi(0)\| + M \int_0^t \|F(s, x_s(\phi : u))\|ds + M\|u\|\sqrt{T}.$$

Thus we have

$$\begin{aligned}&\|x_t(\phi : u)\|_C\\ \leq &M\|\phi(0)\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds + M\|u\|\sqrt{T}.\end{aligned}$$

By hypothesis,

$$\begin{aligned}&\|F(t, x_t(\phi : u))\| \leq \sigma(\|x_t(\phi : u)\|_C)\\ \leq &\sigma(M\|\phi(0)\|_C + M\int_0^t \|F(s, x_s(\phi : u))\|ds + M\|u\|\sqrt{T}).\end{aligned}$$

We get

$$\frac{\|F(t, x_t(\phi : u))\|}{\sigma(M\|\phi(0)\|_C + M\int_0^t \|F(s, x_s(\phi : u))\|ds + M\|u\|\sqrt{T})} \leq 1.$$

Since $\sigma(\cdot)$ is monotonically nondecreasing, $\sigma'(\cdot) \geq 0$. Multiplying both side of the above inequality with an appropriate nonnegative factor, we conclude that

$$\begin{aligned}&\frac{\sigma'(M\|\phi(0)\|_C + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds)M\|F(t, x_t(\phi : u))\|}{\sigma(M\|\phi(0)\|_C + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds)}\\ &\leq M\sigma'(M\|\phi(0)\|_C + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds).\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{\sigma'(M\|\phi(0)\|_C + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds)M\|F(t, x_t(\phi : u))\|}{\sigma(M\|\phi(0)\|_C + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds)}\\ &\leq MM_\sigma,\end{aligned}$$

where $M_\sigma = \sup\{\sigma'(r); r \in R^+\} < \infty$. Integrate both side on $[0, t]$, $t < t_{max} < T$, we arrive

$$\begin{aligned}&\log\left(\frac{\sigma(M\|\phi(0)\| + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi : u))\|ds)}{\sigma(M\|\phi(0)\| + M\sqrt{T}\|u\|)}\right)\\ \leq &MM_\sigma T.\end{aligned}$$

Consequently, we see that

$$\begin{aligned}&\sigma(M\|\phi(0)\| + M\sqrt{T}\|u\| + M\int_0^t \|F(s, x_s(\phi(0))\|ds)\\ \leq&\sigma(M\|\phi(0)\| + M\sqrt{T}\|u\|)e^{MM_\sigma T}.\end{aligned}$$

LEMMA 2. *Let $\phi : R \to R$ be a continuously differentiable monotonically nondecreasing function such that $\lim_{r\to\infty}\phi(r)/r = c$. For every $\tilde{w} \in \tilde{X}$, there exists $\tilde{z} \in \tilde{X}$ such that $\tilde{\mathcal{F}}\tilde{z} = \tilde{w}$ if $c < 1/(MT\|PG\|e^{MM_\sigma T})$.*

Proof. For convenience, let us define $J : \tilde{X} \to \tilde{X}$ by $J = \tilde{\mathcal{F}}(\tilde{u}) - \tilde{w}$, and $B_r = \{\tilde{u} \in \tilde{X} : \|\tilde{u}\| \leq r\}$ for all $r \in R$. Since $\tilde{\mathcal{F}}$ is compact, J is compact. We note that with the chosen magnitude of c,

$$\begin{aligned}&\lim_{r\to\infty}\frac{\sqrt{T}\phi(M\|\phi(0)\| + M\sqrt{T}\|PG\|r)e^{MM_\sigma T}}{r}\\ =&cMT\|PG\|e^{MM_\sigma T} < 1.\end{aligned}$$

Hence, for sufficiently large r_0, we have

$$r_0 \geq \sqrt{T}\phi(M\|\phi(0)\| + M\sqrt{T}\|PG\|r_0)e^{MM_\sigma T} + \|\tilde{w}\|.$$

Thus, for every $\tilde{u} \in B_{r_0}$,

$$\begin{aligned}r_0 &\geq \sqrt{T}\phi(M\|\phi(0)\| + M\sqrt{T}\|PG\|\|\tilde{u}\|)e^{MM_\sigma T} + \|\tilde{w}\|\\ &\geq \sqrt{T}\phi(M\|\phi(0)\| + M\sqrt{T}\|PG\tilde{u}\|)e^{MM_\sigma T} + \|\tilde{w}\|\\ &\geq \|\mathcal{F}(PG\tilde{u})\| + \|\tilde{w}\| \qquad \text{(from Lemma 1)}\\ &\geq \|\tilde{\mathcal{F}}\tilde{u}\| + \|\tilde{w}\|.\end{aligned}$$

Thus, J maps B_{r_0} into itself. So, fixed point of J exists by the Schauder's fixed point theorem, i.e.,

$$\exists\ \tilde{z}\ \text{ such that }\ \tilde{\mathcal{F}}(\tilde{z}) - \tilde{w} = \tilde{z}.$$

THEOREM 2. *Assume that* $F : [0,T] \times C \to X$ *is continuous in* t, *locally Lipschitz in* $\psi \in C$, *uniformly in* $t \in [0,T]$ *and* $\lim_{\|\psi\| \to \infty} \|F(t,\psi)\|/\|\psi\| = c$ *uniformly in* $t \in [0,T]$. *Furthermore, assume that hypotheses (H1) and (H2) are satisfied, then, we have*

$$\overline{K(0)} \subset \overline{K(F)}.$$

Proof. First we show that $K(0) \subset \overline{K(F)}$. For each $\eta \in K(0)$,

$$\eta \in U(t,0)\phi(0) + \int_0^t U(t,s)Bu(s)ds,$$

for some $u(\cdot) \in L^2(0,T : U)$. Let $w = Bu$, $\tilde{w} = w + N$ and $\tilde{z}$ be such that $\mathcal{F}(PG\tilde{z}) + \tilde{z} = \tilde{w}$ (such a $\tilde{z}$ exists by the proof of Lemma 2). Thus, if we let $z = pG(\tilde{z})$, then we have $\tilde{w} = \mathcal{F}(z) + z + N$. From, this, it follows that

$$\begin{aligned}\eta &= U(t,0)\phi(0) + \int_0^t U(t,s)(F(s,x_s(\phi : u)) + z(s))ds \\ &= x_t(\phi : u).\end{aligned}$$

Because $z \in \overline{R(B)}$, there exists a sequence $\{u_n\}$ such that $Bu_n \to z$ as $n \to \infty$. Since $W : L^2(0,T : X) \to C(0,T : C)$ is compact and hence continuous, $W(Bu_n) \to Wz$ in $C(0,T : C)$. Thus $x(t, Bu_n) \to x_t(\phi : u) = \eta$, that is $\eta \in \overline{K(F)}$. Hence $K(0) \subset \overline{K(F)}$. Consequently, we have $\overline{K(0)} \subset \overline{K(F)}$.

References

1. Aubin,J. P., and Ekeland, I., *Applied Nonlinear Analysis*, John Wiley and Sons, New York, 1984.
2. Kwun,Y. C., Park,J. Y.,and Ryu, J. W., *Approximate controllability and controllability for delay volterra system*, Bull. Korean Math. Soc. **28, 2** (1991), 131-45.
3. Naito,K., *Approximate controllability for Trajectories of semilinear control systems*, JOTA **60** (1989), 57-65.
4. Naito, K, *Controllability of semilinear control systems*, SIAM J. on Control and Optimization **25** (1987), 715-722.

5. Naito, K., and Park, J. Y, *Approximate Controllability for Trajectories of a delay Volterra control system*, JOTA **61, 2** (1989), 271-279.

DEPT OF MATH. DONG-A UNIV PUSAN 604-714, KOREA

DEPT OF MATH. PUSAN NATIONAL UNIVERSITY, PUSAN 609-735, KOREA

DEPT OF MATH. DONG-A UNIVERSITY, PUSAN 604-714, KOREA