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A Dual Problem and Duality Theorems for Average Shadow Prices in Mathematical Programming†

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Abstract

Recently a new concept of shadow prices, called average shadow price, has been developed. This paper provides a dual problem and the corresponding duality theorems justifying this new shadow price. The general duality framework is used. As an important secondary result, a new reduced class of price function, the *p.h.-class*, has been developed for the general duality theory. This should be distinguished from other known reductions achived in some specific areas of mathematical programming, in that it sustains the strong duality property in all the mathematical programs. The new general dual problem suggested with this p.h.-class provides, as an optimal solution, the average shadow prices.

1. Introduction

Recently a new concept of shadow prices has been introduced in integer programming [4], and extended to general mathematical programming [2], which has been called the *average shadow price* [2]. It has come from an average analysis which is new and seems the first systematic attempt to depart from the long-standing marginal (shadow) price. This paper provides a new duality theory justfying these average shadow prices. We have used the general duality framework developed by Tind and Wolsey [7]. Strong duality results have been obtained in general mathematical programming models.

Consider the following optimization problem (P).

$$\text{(P)} \qquad \alpha = \sup\{f(x) \mid g(x) \le b,\ x \in X\} \tag{1}$$

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where $f : R^n$ (the Euclidean n-space) $\rightarrow R$, $g : R^n \rightarrow R^m$, and X is a nonempty subset of R^n. We suppose α is finite all through the paper and interpret $g(x) \leq b$ as resource constraints where $b \in R^m$ denotes the current amounts available for the activities x. Define

$$V^0 = \{v : R^m \rightarrow [-\infty, +\infty] \mid v(d_1) \leq v(d_2) \ \forall d_1, d_2 \text{ with } d_1 \leq d_2\}.$$

We consider a subset V of V^0, an element of which is called *a price function* [7] on R^m. The general dual problem (D) with respect to V is defined as

$$\text{(D)} \quad \beta = \inf \{v(b) \mid v(g(x)) \geq f(x) \ \forall x \in X, \ v \in V\} \tag{2}$$

The following fundamental duality results were established.

Theorem 1.1 (*Weak Duality*) [7] If x is primal feasible and v is dual feasible, then $f(x) \leq v(b)$.

Theorem 1.2 (*Strong Duality*) [7] The strong duality holds in the sense that

$$\alpha = \beta = v^*(b)$$

for some feasible price function $v^* \in V$ if $V = V^0$.

The above two theorems, dual feasiblility in (2), and the complementary slackness results derived involve the same useful economic interpretations for resource allocation problems [7] as those involved by the dual prices (multipliers) in linear programming [5].

In spite of the charming economic interpretations we have been without any definite idea of what the optimal price function would be like. The fact is that the strong duality result, valid for all mathematical programs, might hold only under quite an infantile hypothesis $V = V^0$ (**Theorems 1.2**).

It is, out of doubt, worth attempting to reduce V into a smaller class on the condition that it will not violate the strong duality property. However, such reductions have been achieved only in a few local areas of mathematical programming. For example V have been reduced to the set of super-additive price functions [1,3,8] in integer programming. From the existence of subgradients [6] of convex function, V could be reduced to the set of affine functions for a class of nonlinear programming problems including stable convex programming problems [7]. Unlike these previous studies, the suggested class V in this paper sustains the strong duality property throughout all the mathematical programming models. We call this class V the *p.h.* (*positively homogeneous*)*-class*. We also show that the general dual problem (D) with the p.h.-class provides the average shadow prices as an optimal solution. In brief, the duality results obtained in this paper provides a new dual problem which assesses the value of resources in average sense, not in marginal sense.

The next sectin reviews and interprets the concept of ***average shadow price*** and ***average price function***. The duality results, primarily motivated by the fundamental properties of the average price function, are provided in section 3.

2. Average Price Function

The concept of average shadow prices in this section has been motivated by the fact that, in many nonconvex programming models, marginal shadow prices lack practical implications for decision making problems. More of the motivation, development, computation methods and other details are found in [2].

Suppose (P) is a profit maximization model of a firm manager. He may attempt to make some additional profit by changing the current state of resources b along a direction $\rho \in R^m$. For example, $\rho = e_i$, the i-th unit vector in R^m, if he would consider buying a certain additional amount of resource i. If he would consider selling some resource i, ρ would be $-e_i$. He might well want to have some sense, in advance of any real actions, whether the change along ρ is promising.

Definition 2.1. We define the **average shadow price** $p(\rho)$ for an activity $\rho \in R^m$ as

$$p(\rho) = \sup_{t>0} \frac{\phi(b+t\rho)-\phi(b)}{t} \tag{3}$$

where $\phi(d) = \sup\{f(x) \mid g(x) \leq d,\ x \in X\}$ is the perturbation function for (P) with respect to the right hand sides. The function $p: R^m \rightarrow [-\infty, +\infty]$ *is called the* **average price function**.

Fig. 1 shows a typical contrast between the average shadow price $p(\rho)$ and the marginal shadow price (***msp***) where the perturbation function is not concave. $p(\rho)$ measures the contribution of resources in global sense in contrast to the traditional marginal price. In fact, it provides the ***maximum unit profitability*** of resources along a given direction ρ.

Let $\gamma \in R^m$ be the market price vector for the resources and $\langle \cdot , \cdot \rangle$ denote the inner product on the Euclidean space. Then $p(\rho)$ naturally involves a useful suggestion for the decision mak

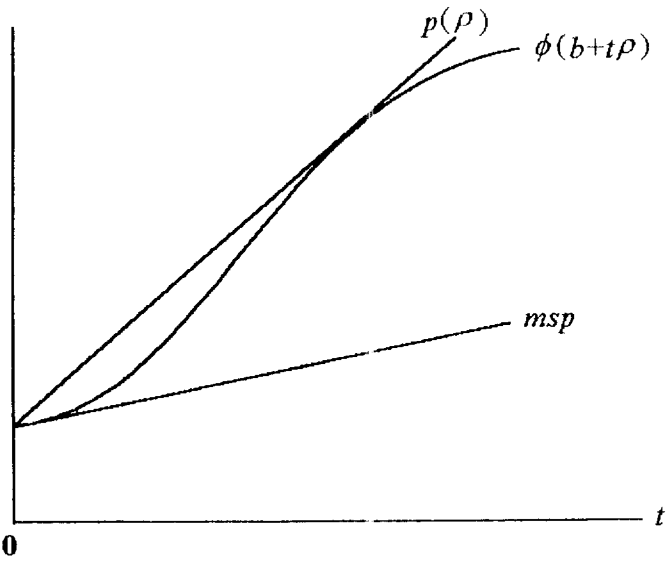

Figure 1. Average shadow price and marginal shadow price

ing problems as follows, which identifies itself in economic sense. Thus knowing $p(\rho)$ helps decide whether to change along ρ or not.

Remark 2.1. If $<\gamma, \rho> < p(\rho)$, then it is ***better to change along*** ρ to a certain extent (since $\phi(b+\bar{t}\rho)-\phi(b)>\bar{t}<\gamma,\rho>$ for some $\bar{t}>0$). If $<\gamma,\rho> > p(\rho)$, then it is ***better not to change along*** ρ (since $\phi(b+t\rho)-\phi(b)<t<\gamma,\rho>$ for any $t>0$).

Theorem 2.1 [2] The average price function $p(\cdot)$ has the following properties.

(a) $p(0)=0$.

(b) $p(\cdot)$ is nondercreasing, i.e., $p(\rho_1)\geq p(\rho_2)$ if $\rho_1\geq\rho_2$.

(c) $p(\cdot)$ is positively homogeneous, i.e., $p(\lambda\rho)=\lambda p(\rho)$ for any $\rho\in R^m$ and positive number λ.

Remark 2.2. From (**c**) of the above proposition the average shadow prices only have to be computed on the unit sphere of $\{\rho\in R^m \mid \|\rho\|=1\}$.

The average shadow prices introduced in this section seem to have practical advantages over the marginal shadow prices for decision making problems, in particular, where nonconvexities are found. It should be noted that the decision criterion as described in Remark 2.1 is not always true to marginal shadow prices. A detailed example can be found in [2, Example 2.1].

The average shadow price was first introduced in integer programming [4]. It is quite a new concept of price, but it also has a universal property that it coincides just with the marginal shadow price in convex programming. It is clear from the fact that $(\phi(b+t\rho)-\phi(b))/t$ is a nonincreasing function of t when $\phi(\cdot)$ is concave.

3. Dual Problem and Duality Results

The following is a generalization of positively homogeneous functions.

Definition 3.1. A function $v: R^m \rightarrow [-\infty, +\infty]$ is said to be **positively homogeneous (p.h.) at** $\bar{d}$ if $v(\bar{d})$ is finite and $v'(d)=v(\bar{d}+d)-v(\bar{d})$ is positively homogeneous (at 0). i.e., $v'(\lambda d)=\lambda v'(d)$ for all $d \in R^m$ and $\lambda > 0$.

In geometric sense ***positive homogeneity at*** $\bar{d}$ of a function $v(\cdot)$ implies that the ***translate of the epigraph*** of $v(\cdot)$ (i.e., the set $\{(d,d_0) \in R^{m+1}\ d_0 \geq v(d)\}$ [6, p.23]), by $(-\bar{d}, -v(\bar{d}))$, is a ***cone.***

Definition 3.2. Let $V(\bar{d})$ is the set of all price functions which are p.h. at $\bar{d}$. The **p.h.-class** of price functions on R^m, denoted by V^1, is defined as the collection of $V(\bar{d})$ as follows.

$$V^1 = \bigcup_{\bar{d} \in R^m} V(\bar{d}) \tag{4}$$

Remark 3.1. Let $V=V^1$. Then it is easy to see that V is a *cone* and closed under number addition. In other words for any positive real constant c,

$$cv \in V \quad \text{if } v \in V \tag{5}$$

and for any real number h

$$v+h \in V \quad \text{if } v \in V. \tag{6}$$

It has been proved [7, **Proposition 5.8**] that (D) in (2) is equivalent to the generalized Lagrangean dual problem if V satisfies the condition (6).

Theorem 3. 1. Consider the primal-dual pair (P) and (D) of (1), (2) with $V=V^1$. Then (D) is a *strong dual problem* of (P) in the sense that there exists a feasible price function v^* of (D) with $v^*(b)=\alpha$.

Proof. Define

$$v^*(d)=\alpha+p(d-b) \tag{7}$$

where $p(\cdot)$ is the average price function in **Definition 2.1**. Then $v^*(b)=\phi(b)=\alpha$ which is finite. We only have to show that v^* is a feasible price function. It is clear that v^* is nondecreasing since $p(\cdot)$ is nondercreasing from Theorem 2.1. Since $p(d)$ is also a positively homogeneous function of d, v^* is p.h. at b. Thus $v^*\in V$. Now choose any $x\in X$, then from (7) and the definition of the average price fuction,

$$\begin{aligned} v^*(g(x)) &= \phi(b)+p(g(x)-b)\geq\phi(b)+\phi(g(x))-\phi(b) \\ &= \phi(g(x))\geq f(x). \end{aligned}$$

We show v^* is feasible and so optimal to (D). ■

The above theorem suggests a way how the strong duality can be kept through all mathematical programs under a much smaller class of price functions than the orginal V.

Remark 3.2 In the proof of the above theorem we have used an optimal price function of (7) which, in fact, becomes the average price function by a simple translate. As **Fig.2** typically shows, (D) with $V=V^1$ always produces the average price function as an optimal solution.

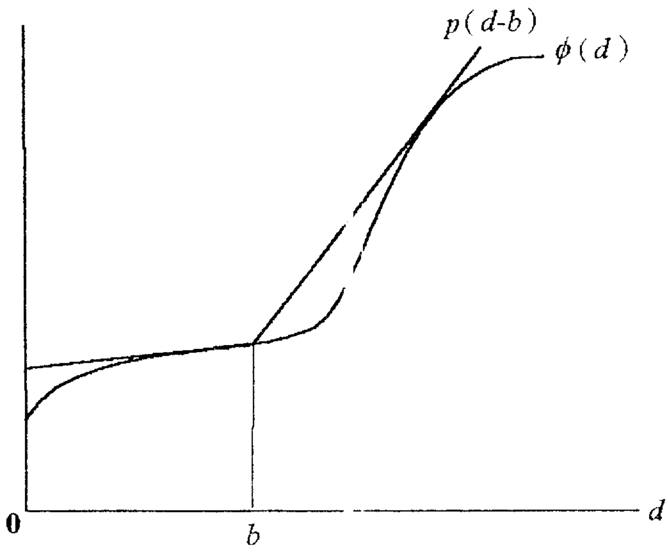

Fig. 2 Average price function as an optimal price function

Theorem 3.1 guarantees a strong duality in every mathematical program. However the achieved optimal price function may appear in a trivial shape. For example, we can easily show that the following trivial price function $v(\cdot)$ is an optimal price function of (D) with $V=V^1$.

$$v(d) = \begin{cases} \phi(b), & \text{if } d \leq b \\ +\infty, & \text{otherwise} \end{cases}$$

To exclude the above trivial solutions, we could say a function $v \in V^1$ is ***improper*** if it somewhere attains $+\infty$, and *proper*, otherwise.

Definition 3.3. We define the **proper p.h.-class** of price functions on R^m as

$$V^2 = \{v \in V^1 \mid v \text{ nowhere attains } +\infty\}. \tag{8}$$

We can also prove a strong duality theorem in the sense that there exist no duality gap only by using the above proper p.h.-class, in case $\phi(\cdot)$ is bounded from above, and upper-semi-continuous (u.s.c.) at b (i.e., $\lim \sup_{d \to b} v(d) = v(b)$ [6, p.51]).

Lemma 3.2. If $\phi(\cdot)$ is u.s.c. at b and bounded from above, then for any given $e>0$,

$$p_e(\rho) = \sup_{t>0} \frac{\phi(b+t\rho) - \phi(b) - e}{t} < +\infty$$

for all $\rho \in R^m$.

Proof. Choose any $\rho \in R^m$ and any $\hat{t}>0$, and consider any $t \geq \hat{t}$. Since ϕ is bounded from above

$$\frac{\phi(b+t\rho) - \phi(b) - e}{t} \begin{cases} = -\infty, & \text{if } (\mathrm{P}(b+t\rho)) \text{ is infeasible} \\ \leq M, & \text{if } (\mathrm{P}(b+t\rho)) \text{ is feasible} \end{cases}$$

for some positive number M. Hence

$$\sup_{t \geq \hat{t}} \frac{\phi(b+t\rho) - \phi(b) - e}{t} < +\infty$$

for any $\hat{t}>0$. From the u.s.c. property of $\phi(\cdot)$ at b it is clear that $\phi(b+t\rho) - \phi(b) < e$ for all sufficiently small numbers t. This leads to $p_e(\rho) < +\infty$. ■

Remark 3.3. Note that the function $\rho \to p_e(\rho)$ is also positively homogeneous, nondecreasing, and $p_e(0)=0$. It is also clear that $p_e=-\infty$ if and only if $p(\rho)=-\infty$.

The u.s.c.-property of $\phi(\cdot)$ is essential for the above lemma to hold as is clear from the following simple example.

Example 3.1. Consider the following problem.

$$(P)\quad \alpha=\max\{x_2 \mid -x_1+x_2\leq 0,\ (x_1,x_2)\in X\}$$

where $X=\{(x_1,x_2) \mid 0\leq x_1<1,\ x_2=0 \text{ or } 1\}$. The perturbation function $\phi(\cdot)$ is not u.s.c. at 0, since $\phi(0)=0$ but $\phi(d)=1\ \forall d>0$. **Lemma 3.2** no longer holds since $p_e(\rho)=\sup_{t>0}((1-e)/t)=+\infty$ for $\rho=1$ and for any e with $0<e<1$.

Theorem 3.3. If $\phi(\cdot)$ is bounded from above and u.s.c. at b, then the strong duality holds between (P) andd (D) with $V=V^2$, in the sense that no duality gap exists between the primal-dual pair. That is,

$$\alpha=\beta.$$

Proof. Choose any $e>0$. Now define

$$v_e(d)=\alpha+e+p_e(d-b).$$

Then it is clear that $v_e\in V$ from **Lemma 3.2** and **Remark 3.3.** The feasibility of v_e can be shown in the similar way to the proof of **Theorem 3.1.** Moreover we see

$$v_e(b)-\alpha=e.$$

Since e is arbitrarily chosen we have proved that $\alpha=\beta$. ■

Example 3.2. Consider the following simple problem.

$$(P)\quad \alpha=\max\{\sqrt{x} \mid x\leq 0,\ x\in[0,1]\}$$

It is clear that $\phi(\cdot)$ is u.s.c. at $b=0$ and bounded from above. But we cannot find any optimal price function for the general dual problem (D) with $V=V^2$. However we can find a following sequence of ***proper*** feasible price functions $\{v_n$ where $v_n(d)=1/n+(n/4)d$. It is easy to see that the above v_n is feasible and that $\lim_{n\to\infty} v_n(\ )=0=\alpha$. Note, however, that $\lim_{n\to\infty} v_n(\cdot)$ is ***im-proper*** and therefore infeasible.

Suppose the average price function is finite on R^m, for example, this holds if $\phi(\cdot)$ is bounded from above, and ***locally Lipschitz continuous*** at b (i.e., for some positive constant L, we can find a neighborhood N of b where $|\phi(d_1)-\phi(d_2)|\leq L\|(d_1-d_2\|$ for any $d_1\in N$, $d_2\in N$). Then we may again reduce the proper p.h.-class to consist of only real valued price functions. That is,

$V^3=\{v\in V^2|v$ is real valued$\}$

is enough to prove **Theorem 3.1**. The simplest type of proper price functions in the p.h.-class would be affine functions. (It is clear that proper p.h.-class is far larger than the set of affine price functions.) The existence of an affine function as an optimal price function, however, is equivalent to that $\phi(\cdot)$ is ***subdifferentiable*** at b [6, p.215], i.e.,

$\phi(d)\leq\phi(b)+\langle\gamma,d-b\rangle$ for all $d\in R^m$.

as was already mentioned [7, **Theorem 8.1-2**].

4. Conclusions

This paper has developed a new dual problem for the average shadow prices. The strong duality results are achieved, and the average shadow prices are found as an optimal price function.

With regard to the theory of general duality, also obtained are three improved results. ***First***, a great reduction of price functions, the p.h.-class, has been found with which the strong duality property is kept through all the mathematical programming models. This reduction should be distinguished from other known results, which have been valid only in some specific classes of mathematical programming.

Second, the new general dual problem derived from the p.h.-class provides the average price function as an optimal price function. The average price function measures the values of resources in an average sense in contrast to the traditonal marginal prices and suggests useful information for decision making problems about resources (**Section 2**). The conclustion is that we have made up a new dual problem using the general duality framework, which sustains the strong duality in all the mathematical programs, and the dual problem yields, as an optimal solution, the average price function which assesses the value of resources in average sense.

Finally, the p.h.-class is a first prior reduction of the price functions valid for all the mathematical programs. Still we might be able to re-reduce the p.h.-class in some specific classes of mathematical programming. This must be one of the major future researches and started from identifying more refined properties of the average price function involved with these specific areas. For example, the p.h.-class can be again reduced to the set of affine functions in stable convex programming, which is clear for $\phi(\cdot)$ is subdifferentiable at b.

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