On the Properties of Scaling Exponents for the Dissipative System

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消耗系에서 縮尺指數의 性質에 관한 考察

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Wilbrink 본뜨기에서 모우드 라킹 현상과 소모적 궤적의 두 경우에 대한 일반화차원 D_n을 수치해석적으로 계산하였다. 투닝변수 $z=0.03$, 소모변수 $b=0.9$, $k_d=0.272313668$의 값으로 주어진 소모적 Wilbrink 본뜨기에서 모우드 라킹현상의 경우에는 $n\sim20$일때 $D_{-20}=0.92402$의 값을 갖으며, 소모적궤적에서는 $D_{-20}=0.63292$와 $D_{+20}=1.89877$의 값으로 주어진다. 이때의 값들은 $n\rightarrow\infty$감에 따라 Circle 본뜨기의 $D_{\pm\infty}$값들과 근사적으로 일치한다.

1. Introduction

Recently, the chaotic behavior has been exhibited by an strange attractor[1,2] on phase space in a dissipative dynamic system. Strange attractors are typically characterized by a non-integer exponent called fractal dimension[3]. Considering an infinite set of fractal dimensions, we can extend to the distribution of a kind of singularity of the measure associated with multifractals[4-6] and the important property of multifractals has a spectrum of scaling exponents.

The concept of multifractals has been introduced by Paladi et al[7], realizing that the moment scaling exponents can be related to the scaling of probability distribution of the singularities. The multifractal object can be regarded in this approach, further developed by Halsey et al[6]. Until now, multifractals have appeared in a large field of physical problems, such as chaos in dynamic system, turbulence, percolating cluster and their backbones, diffusion-limited aggregation[8,9], random resistor networks, and mass multifractals[10].

In the general case of chaotic behavior, the generalized dimensions D_n[10,11] is derived as

$$D_n=(1-n)^{-1}(n\alpha(n)-f(\alpha(n))) \quad (1)$$

In eq.(1) D_n shows a measure of inhomogeneity in the probability distribution on the attractor, and the measure on the attractor has described by the interwoven sets with the fractal dimensionality $f(\alpha)$.

The principal purpose of this paper is to carry out by both analytical and numerical techniques on the generalized dimensions D_n and the fractal dimensionality $f(\alpha)$ in the mode-locking phenomenon and the dissipative trajectory for a dissipative standardlike map of Wilbrink[12,13].

This paper is organized as follows. In Sec.2 we describe the formula related between D_n and $f(\alpha)$. In Sec. 3 the values of D_n and $f(\alpha)$ will be estimated and discussed for a special example of dissipative Wilbrink map. In Sec. 4 a brief discussion and summary will be given.

2. Generalized dimension and fractal dimensionality

Concentrating on the probaility for points of strange attractors falling within i-th box of size l in a phase space, this probability can be described as

$$P_i(l)=l^{\alpha} \quad (2)$$

where the scaling exponents α takes $\alpha_{\min}<\alpha<\alpha_{\max}$ for small l. If the system is divided into pieces of size l, one finds the number of times, $N_\alpha(l)$, falling in interval of size $d\alpha$ via

$$N_\alpha(l)=d\alpha\rho(\alpha)l^{-f(\alpha)} \quad (3)$$

The fractal dimensionality $f(\alpha)$ can be interpreted as the dimensions of the subsets with scaling exponent α. Now we can relate $f(\alpha)$ to the partition function $\Gamma(n, l)$, i.e., the moments of probability $P_i(l)$. Following eqs. (2) and (3) one has

$$\Gamma(n,l)=<p_i(l)^n>\propto\int d\alpha\ d\rho(\alpha)l^{\alpha n-f(\alpha)} \quad (4)$$

We have showed a set of generalized scaling exponents :

$$\Gamma(n,1)=<p_i(l)^n>=<m_i^{-n}> \quad (5)$$

So that we estimate $P_i(l)$ as the inverse recurrence time, m_i^{-1}, and find by eq.(2)

$$\alpha=-\ln m_i/\ln l \quad (6)$$

Also, the partition fuction of eq.(4) for $l \ll 1$, is given by a power l,

$$\Gamma(n,l)\propto l^{\tau(n)} \quad (7)$$

The quantity $\tau(n)$[14] is related to the generalized dimensions via Legendre transformation relation

$$\tau(n)=(n-1)D_n \quad (8)$$

It then follows from eq.(4) and eq.(8) that

$$(n-1)D_n=[n\alpha(n)-f(\alpha(n))] \quad (9)$$

Thus, if we know $f(\alpha)$, then we can find D_n and, alternatively, given D_n, we can find $\alpha(n)$ by the relation

$$\alpha(n)=\frac{d}{dn}[(n-1)D_n] \quad (10)$$

3. Fractal dimensionality for Wilbrink map

Wilbrink[15] has studied the special case of a two-parameter standard map as

$$r_{n+1}=br_n+kg(\theta_n)$$
$$\theta_{n+1}=\theta_n+\Omega+r_{n+1} \quad (11)$$

where

$$g(\theta_n)=-\frac{\sqrt{1+z}}{2\pi}\arcsin\left(\frac{\sin(2\pi\theta n)}{\sqrt{1+z}}\right) \quad (12)$$

This map is among the standard map for the area-preserving case, $b=1$, and it reduces the sine-circle map for the infinitely dissipative case, $b=0$. In case of $b=1$, In the case of b=1, Wilbrink has found that the invariant circles can reappear for small value z. This case is in contrast to the standard map where there is no reappearance of invariant circles. Kim and Hu[12] has studied the transition between the conservative and dissipative cases by varying the dissipative parmeter from $b=0$ to $b=1$. They have shown the breakup of an invariant circle whose rotation number is the reciprocal golden mean. When both the tuning parameter z and the dissipation parameter b are small in the dissipative Willbrink map, it has been found that the invariant circle can reappear after it disappeared when the nonlinearity is increased. Especially, when $b\leq b^*=0.65$, there is no recurrence of invariant circles.

Next, we introduce the convergence ratios $\delta_n(k)$ and α_n^* in order to relate to the generlaized dimension D_n. the convergence ratio $\delta_n(k)$ of the sequence $\Omega_n(k)$[16,17] at which the residue of the periodic orbit with rotation number w_n has its maximum value is defined as

$$\delta_n(k)=\frac{\Omega_{n-1}(k)-\Omega_n(k)}{\Omega_n(k)-\Omega_{n+1}(k)} \quad (13)$$

where $W_n=F_n/F_{n-1}$ and F_n is the n-th Fibonacci number($F_{n+1}=F_n+F_{n-1}$ with $F_0=0$ and $F_1=1$)

Define the convergence ratio α_n^* by

$$\alpha_n^*=\frac{d_{n-1}(k)}{d_n(k)} \quad (14)$$

where d_n is the angular distance

$$d_n=\theta_{Fn}-\theta_0-F_{n-1}. \quad (15)$$

For the special example of $z=0.03$, $b=0.9$ and $k_d=0.272313668$ in Wilbrink map, we will find the scaling relations in the most rarefied and concentrated region near $\theta_0=0$. When $b^*=0.65\leq b=0.09<1$, this case indicates that the resonances of the golden mean invariant circle can be separated after they overlap. The value of k_d is that of k existing the first disappearance point where the invariant circle is broken.

To calculate the generalized dimensions D_n and the Fractal Dimensionality $f(\alpha)$ analytically, we consider both the mode-locking phenomenon[18,19] and the dissipative trajectory for Wilbrink map.

Mode-locking Phenomenon. The mode-locking structure for dynamic system occurs mainly when the bare winding number $\Omega=w_1/w_2$ is close to a rational number. It is also estimated that the dressing winding number $w_n=F_n/F_{n+1}$ is constant and rational for small value Ω. Until now the mode-locking structure has well known in terms of a complete de-

Table 1. Value of δ_n[12], δ_n^* and D_{-n} on Wilbrink map for mode-locking phenomenon when $z=0.003$, $b=0.9$, and $k_d=0.272313668$.

n	δ_n	δ_n^*	D_{-n}
5	-2.63485	2.01330	0.99339
6	-2.78646	2.12956	0.93915
7	-2.84036	2.16937	0.92192
8	-2.83740	2.16721	0.92284
9	-2.83798	2.16763	0.92266
10	-2.83775	2.16746	0.92273
11	-2.83760	2.16735	0.92278
12	-2.83608	2.16624	0.92325
13	-2.83511	2.16553	0.92355
14	-2.83436	2.16498	0.92379
15	-2.83400	2.16472	0.92390
16	-2.83378	2.16455	0.92397
17	-2.83370	2.16450	0.92400
18	-2.83365	2.16446	0.92401
19	-2.83363	2.16444	0.92402
20	-2.83362	2.16444	0.92402
.	.	.	.
.	.	.	.
.	.	.	.
∞	-2.83360	2.16442	0.92403

Table 2. Values of α_n^*[20], D_n, and D_{-n} on Wibrink map for dissipative trajectory when $z=0.03$, $b=0.9$, and $k_d=0.272313668$.

n	α_n^*	D_{+n}	D_{-n}
5	-1.55842	0.36153	1.08461
6	-1.54598	0.36818	1.10456
7	-1.37303	0.50597	1.51792
8	-1.36234	0.51876	1.55629
9	-1.38289	0.49480	1.48441
10	-1.29906	0.61306	1.83920
11	-1.29222	0.62569	1.87708
12	-1.28873	0.63236	1.89709
13	-1.28779	0.63418	1.90256
14	-1.28760	0.43455	1.90367
15	-1.28771	0.63434	1.90303
16	-1.28791	0.63395	1.90186
17	-1.28810	0.63358	1.90075
18	-1.28826	0.63327	1.89982
19	-1.28837	0.63306	1.89918
20	-1.28844	0.63292	1.89877
.	.	.	.
.	.	.	.
.	.	.	.
∞	-1.28857	0.63267	1.89802

vil's staircase representing the dressed winding number[18,19].

From now on we calculate analytically the generalized dimension D_n and use eq.(9) and eq.(10) to extract $f(\alpha)$. The following expressions for D_{-n} in the most rarefied region and D_{+n} in the most concentrated region are given by

$$D_{-n}=\frac{\ln P_{-n}}{\ln l_{-n}}=\frac{2}{\delta_n^*} \tag{16}$$

$$D_{+n}=\frac{\ln P_n}{\ln l_n}=\frac{1}{2} \tag{17}$$

In the first equality of eq.(16) the probability P_{-n} and the length scale l_{-n} are respectively proportional to $\bar{w}^{n\delta^*}$ and $\bar{w}^{2n}$. The value of reciprocal of golden mean $\bar{w}$ has(5 $-1)/2$[16]. The values of generalized dimension D_{-n} in eq.(16) are listed in Table. 1. In particular, the value D_n is obtained from eq.(17) as $P_n \sim l_n^{1/2}$ in the most concetrated region. The maximum balue of $f(\alpha)$, i.e., the fractal dimension D_0, gives as $D_0=0.871$ in mode-locking structure[13]. This value is in good agreement with the results of Refs. (18) and (19).

Dissipative Trajectory for Wilbrink map. In the dissipative Wilbrink map, it exists the transition between the conservative and dissipative cases by varying the dissipation parameter value from $b=0$ to $b=1$. As $b \rightarrow 0$, it is obvious that the critical points for this case and the circle map converge to the first disappearance point k_d[12].

In dissipative trajectory the generalized dimension D_n can be written as

$$D_{-n}=\frac{\ln P_{-n}}{\ln l_{-n}}=\frac{\ln \overline{w}}{\ln \alpha_n^{*-1}} \quad (18)$$

$$D_n=\frac{\ln P_n}{\ln l_n}=\frac{\ln \overline{w}}{\ln \alpha_n^{*-3}} \quad (19)$$

where $P_{+n}\sim\overline{w}^n$, $l_{\ n}\sim\alpha_n^{*-n}$ and $l_{\pm n}\sim\alpha_n^{*-3n}$.

From eq.(9) and eq.(10), we obtain the value of α given as $\alpha_{max}=D_{-n}$ and $\alpha_{min}=D_{+n}$, when $f(\alpha)=0$ and the values of D_n and D_{-n} are listed in Table 2. the maximum value of $f(\alpha)$ is estimated to $D_0=1$ at the arbitrary values α_n^* as listed in Table 2.

4. Summary

We have investigated analytically and numerically on both the generalized dimension D_n and the fractal dimensionality $f(\alpha)$ in the dissipative Wilbrink map, and discussed both the mode-locking phenomenon and the dissipative trajectory when $z=0.03$, $b=0.9$ and $K_d=0.272313668$.

In the mode-locking phenomenon, we find that the generalized dimension D_{-n} and superconverged δ_n^* are very close to $D_{-\infty}=0.92403$ and $\delta_\infty^*=2.16442$ even for $n\sim 20$ as listed in Table 1. In dissipative trajectory, the values of D_{+n} and D_{-n} for $n\sim 20$ are estimated to be very close to $D_{+\infty}=0.63267$ and $D_{-\infty}=1.89802$ on the circle map[16,17]. Thus, the values of the generalized dimension as $n\to\infty$ on dissipative Wilbrink map are expected to be the same results as those for the circle map and to have the universal scaling exponents for a special scaling structure when the values of $\overline{w}$, z, b, and k_d have the different values.

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