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Sufficient Conditions for the Admissibility of Estimators in the Multiparameter Exponential Family

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ABSTRACT

Consider the problem of estimating an arbitrary continuous vector function under a weighted quadratic loss in the multiparameter exponential family with the density of the natural form. We first provide, using Blyth's (1951) method, a set of sufficient conditions for the admissibility of (possibly generalized Bayes) estimators and then treat some examples for normal, Poisson, and gamma distributions as applications of the main result.

KEYWORDS: Multiparameter exponential family, Weighted quadratic loss, Generalized Bayes, Admissibility.

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1. INTRODUCTION

Let $X = (X_1, X_2, \cdots, X_p)$ be a random vector on $\mathcal{X} \subset \mathrm{R}^p$ whose density is given by

$$f(x;\theta) = e^{\theta \cdot x - \varphi(\theta)}, \theta = (\theta_1, \cdots, \theta_p) \in \Theta \subset \mathrm{R}^p, x \in \mathcal{X} \subset \mathrm{R}^p \qquad (1.1)$$

with respect to some σ-finite measure μ on $\mathcal{X}$ where Θ is taken to be the natural parameter space, $\Theta = \{\theta : \int e^{\theta \cdot x}\mu(dx) < \infty\}$. Consider the problem of estimating a continuous vector function $\nabla r(\theta)$ under a weighted quadratic loss $L(\theta, d) = \sum_{i=1}^{p} V_i(\theta)(d_i - \nabla_i r(\theta))^2$, where $d = (d_1, \cdots, d_p) \in D \subset \mathrm{R}^p$, the decision space, and $V_i(\theta)$, $i = 1, \cdots, p$, are positive and differentiable functions on Θ.

For general $p(\geq 1)$, Brown and Hwang (1982) have developed a simple and unified approach using Blyth's(1951) method for proving the admissibility of (possibly generalized Bayes) estimators of the mean vector $\nabla\varphi(\theta)$ under a weighted quadratic loss with a single sequence of priors for all estimators in the multiparameter exponential family with the density (1.1). Das Gupta and Sinha(1984), using Brown and Hwang's technique which is in turn based on Blyth's method, gave sufficient conditions for the admissibility of (possibly generalized Bayes) estimators of $\nabla r(\theta)$, other than the mean $\nabla\varphi(\theta)$, under the sum of squared error losses where $r(\theta)$ is a function with continuous partial derivatives. Recently, for $p = 1$ Kim(1991) obtained, using Blyth's method, sufficient conditions of the admissibility of (possibly generalized Bayes) estimators of an arbitrary (piecewise) continuous function $h(\theta)$ under a squared error loss. These sufficient conditions are different from those of Das Gupta and Sinha (1984) for $p = 1$. For $p \geq 1$, Dong(1990) gave sufficient conditions for the admissibility of (possibly generalized Bayes) estimators of $\nabla r(\theta)$ under a weighted quadratic loss.

The purpose of this thesis is to provide sufficient conditions different from those of Dong(1990) using Brown and Hwang's technique for the admissibility of (possibly generalized Bayes) estimators of $\nabla r(\theta)$ in the multiparameter exponential family with the density (1.1) under a weighted quadratic loss. This result partially generalizes results of Brown and Hwang(1982) for $V_i(\theta) = 1$, $i = 1, 2, \cdots, p$, and $\nabla r(\theta) = \nabla\varphi(\theta)$, Das Gupta and Sinha(1984) for $V_i(\theta) = 1$, $i = 1, 2, \cdots, p$, and Kim(1991) for an arbitrary continuous function $h(\theta)$. In Section 2 we treat some preliminaries including Blyth's method which is crucial in our analysis. In Section 3 we give, using Blyth's method, a set of sufficient conditions for the admissibility of (possibly generalized Bayes) estimators of $\nabla r(\theta)$ under a weighted quadratic loss. Finally, Chapter 4 contains some examples for normal, Poisson, and gamma distributions as applications of the main result.

2. PRELIMINARIES

Let X be a random vector with the density (1.1). Consider the problem of estimating $\nabla r(\theta)$, a continuous vector function of θ, under a weighted quadratic loss

$$L(\theta, d) = \sum_{i=1}^{p} V_i(\theta)(d_i - \nabla_i r(\theta))^2, \ \theta \in \Theta, \ d \in D, \tag{2.1}$$

where $V_i(\theta)$, $i = 1, \cdots, p$, are positive and differentiable functions and $\nabla r(\theta) = (\nabla_1 r(\theta), \cdots, \nabla_p r(\theta))$, $\nabla_i r(\theta) = \partial r(\theta)/\partial \theta_i$, $i = 1, 2, \cdots, p$.

The convexity of the loss functon (2.1) permits us to restrict attention only to nonrandomized estimators. See Ferguson(1967, p78) or Berger (1985, p40). Furthermore, there is no loss of generality in restricting our attention to the case of a single observation X for, as is well-known, the vector of the sums of the obervations in a sample of size n from the density (1.1) is sufficient for θ whose distribution also has the density (1.1).

Consider a prior distribution $\Pi(\cdot)$ with the differentiable density $\pi(\cdot)$ with respect to Lebesgue measure. Assume $\Pi(K) < \infty$ for all compact set $K \subset \Theta$, and define for fixed $\eta(\neq -1) \in \mathrm{R}^1$ and $\alpha \in \mathrm{R}^p$,

$$I_x(h) = \int_{\Theta} h(\theta) e^{\theta \cdot (x+\alpha) - (\eta+1) r(\theta)} \, d\theta, x \in \mathcal{X}.$$

Assume, for $i = 1, 2, \cdots, p$,

$$I_x\{|\nabla_i (V_i(\theta) g(\theta))|\} < \infty, x \in \mathcal{X}, \tag{2.2}$$

where $g(\theta) = \pi(\theta) e^{-\alpha\cdot\theta - \varphi(\theta) + (\eta+1) r(\theta)}$ is differentiable everywhere.

Let $\delta_\pi(x)$ have the i^{th} coordinates.

$$\delta_\pi^i(x) = \frac{x_i + \alpha_i}{\eta + 1} + \frac{I_x[\nabla_i(V_i(\theta) g(\theta))]}{(\eta + 1) I_X[V_i(\theta) g(\theta)]}, \quad i = 1, \cdots, p, \tag{2.3}$$

with the obvious convention that $I_x[\nabla_i(V_i(\theta)g(\theta))]/\infty = 0$, $i = 1, \cdots, p$. Note that $I_x(V_i(\theta)g(\theta)) > 0$ for all $x \in \mathcal{X}$, and hence δ_π^i in (2.3) is well defined for $i = 1, \cdots, p$. Throughout this paper assume that $R(\theta, \delta_\pi) < \infty$ for all $\theta \in \Theta$.

Remark 2.1. When $\Theta = \mathrm{R}^p$, if for each $x \in \mathcal{X}$.

$$I_x[V_i(\theta)g(\theta)] < \infty, \text{ for all } i = 1, \cdots, p, \tag{2.4}$$

in addition to (2.2), then $\delta_\pi(X)$ is the generalized Bayes estimator of $\nabla r(\theta)$ with respect to the prior $\Pi(\theta)$. This can be shown as follows: By integration by parts, for $i = 1, \cdots, p$,

$$\begin{aligned} & I_x[\nabla_i(V_i(\theta)g(\theta))] = \int_{\mathrm{R}^p} \nabla_i(V_i(\theta)g(\theta))e^{\theta\cdot(x+\alpha)-(\eta+1)r(\theta)}\, d\theta \\ = \; & -(x_i + \alpha_i)I_x(V_i(\theta)g(\theta)) + (\eta+1)I_x[(V_i(\theta)g(\theta))(\nabla_i r(\theta))]. \end{aligned} \tag{2.5}$$

Hence if (2.2) and (2.4) hold, we have, from (2.5),

$$\delta_\pi^i(x) = \frac{x_i + \alpha_i}{\eta + 1} + \frac{I_x[\nabla_i(V_i(\theta)g(\theta))]}{(\eta+1)I_x(V_i(\theta)g(\theta))} = \frac{I_x[V_i(\theta)g(\theta)\nabla_i r(\theta)]}{I_x[V_i(\theta)g(\theta)]}.$$

On the other hand, when $\Theta \neq \mathrm{R}^p$, say $\Theta = \overset{p}{\times}_{i=1}(a_i, b_i)$, if

$$\lim_{\theta_i \to {a_i \atop b_i}} V_i(\theta)g(\theta)e^{\theta_i(x_i+\alpha_i)-(\eta+1)r(\theta)} = 0 \text{ for } i = 1, \cdots, p, \tag{2.6}$$

in addition to (2.2) and (2.4), then $\delta_\pi(X)$ in (2.3) is an appropriate generalized Bayes estimator of $\nabla r(\theta)$.

We now introduce the Blyth's(1951) method for providing the admissiblity of estimators, stated below in the form appeared in Berger (1976, p345, Theorem 3). See also Stein(1955), Farrell(1964), and Berger(1985, p547).

Lemma 2.1. Let $\{h_n\}$ be a sequence of absolutely continuous functions defined on Θ satisfying

(1) $\int_\Theta h_n^2(\theta)\pi(\theta)\, d\theta$ for all $n = 1, 2, \cdots$;

(2) for all $n = 1, 2, \cdots, h_n(\theta) \geq K > 0$ for all θ in a set C for which $\int_C \pi(\theta)\, d\theta > 0$;

(3) $h_n(\theta) \to 1$ a.e. (Lebesgue measure) as $n \to \infty$.

Consider a sequence $\{\pi_n\}$ of prior densities with respect to Lebesgue measure such that $\pi_n(\theta) = h_n^2(\theta)\pi(\theta)$, $n = 1, 2, \cdots$. Then if

$$\boldsymbol{\Delta} \equiv \int_{\boldsymbol{\Theta}} [\mathbf{R}(\theta, \delta_\pi) - \mathbf{R}(\theta, \delta_{\pi_\mathbf{n}})]\pi_\mathbf{n}(\theta)\, \mathbf{d}\theta \to \mathbf{0} \textbf{ as } \mathbf{n} \to \infty,$$

then $\delta_\pi(X)$ is admissible where δ_{π_n} is the corresponding Bayes estimator with respect to the proper prior $\pi_n(\theta)$.

Remark 2.2. When $\Theta = \mathsf{R}^p$, if for all $x \in \mathcal{X}$ and $i = 1, \cdots, p$,

$$I_x[V_i(\theta)h_n^2(\theta)g(\theta)] < \infty, \tag{2.7}$$

and

$$I_x[|\nabla_i(V_i(\theta)h_n^2(\theta)g(\theta))|] < \infty, \tag{2.8}$$

then the Bayes estimator $\delta_{\pi_n}(X)$ with respect to $\pi_n(\theta) = h_n^2(\theta)\pi(\theta)$ under the loss (2.1) has the i^{th} coordinate

$$\delta_{\pi_n}^i(x) = \frac{x_i + \alpha_i}{\eta + 1} + \frac{I_x[\nabla_i(V_i(\theta)h_n^2(\theta)g(\theta))]}{(\eta+1)I_x[V_i(\theta)h_n^2(\theta)g(\theta)]}, \tag{2.9}$$

$i = 1, \cdots, p$. This can be easily seen by using integration by parts. Furthermore, when $\Theta \neq \mathsf{R}^p$, say, $\Theta = \mathop{X}_{i=1}^{p}(a_i, b_i)$, if

$$\lim_{\theta_i \to a_i, b_i} V_i(\theta)h_n^2(\theta)g(\theta)e^{\theta_i(x_i+\alpha_i)-(\eta+1)r(\theta)} = 0, \tag{2.10}$$

in addition to (2.7) and (2.8), then $\delta_{\pi_n}(X)$ in (2.9) is the Bayes estimator with respect to $\pi_n(\theta) = h_n^2(\theta)\pi(\theta)$ under the loss (2.1).

3. SUFFICIENT CONDITIONS FOR ADMISSIBILITY

In this chapter we give the main result which provides sufficient conditions for $\delta_\pi(X)$ in (2.3) to be admissible for estimating $\nabla r(\theta)$ under the loss (2.1).

Theorem 3.1. Let $\Pi(\theta)$ be a prior distribution satisfying (2.2). Assume that there exists a sequence $\{h_n(\theta)\}$ of absolutely continuous functions defined on Θ satisfying the conditions (1),(2), and (3) of Lemma 2.1. and the conditions (2.7),(2.8), and (2.10). Then $\delta_\pi(X)$ in (2.3) is admissible for estimating $\nabla r(\theta)$ under the loss (2.1) if

$$\sum_{i=1}^{p}\int_\Theta[\nabla_i h_n(\theta)]^2 V_i(\theta)\pi(\theta)\,d\theta \to 0 \text{ as } n\to\infty, \tag{3.1}$$

$$\sum_{i=1}^{p}\int_\Theta[\nabla_i\{\ln V_i(\theta)\pi(\theta)\}]^2 V_i(\theta)\pi(\theta)\,d\theta < \infty, \tag{3.2}$$

and

$$\sum_{i=1}^{p}\int_\Theta V_i(\theta)\pi(\theta)[-\alpha_i - \nabla_i\varphi(\theta) + (\eta+1)\nabla_i r(\theta)]^2\,d\theta < \infty. \tag{3.3}$$

Proof. By Lemma 2.1, it is enough to show that

$$\Delta_n = \int_\Theta [R(\theta,\delta_\pi) - R(\theta,\delta_{\pi_n})]\pi_n(\theta)\,d\theta \to 0 \text{ as } n\to\infty,$$

Now,

$$\Delta_n = \sum_{i=1}^{p}\Delta_n^i,$$

where

$$\begin{aligned}\Delta_n^i &= \int_\Theta[\int_{\mathcal{X}} V_i(\theta)[(\delta_\pi^i(x)-\nabla_i r(\theta))^2 - (\delta_{\pi_n}^i(x)-\nabla_i r(\theta))^2]\\ &\quad\cdot f(x;\theta)\mu(dx)]\pi_n(\theta)\,d\theta, \text{ for } i=1,\cdots,p.\end{aligned}$$

Hence, it suffices to show that $\Delta_n^i \to 0$ as $n\to\infty$ for each $i=1,\cdots,p$. Now, applying Fubini's theorem yields

$$\begin{aligned}\Delta_n^i &= \int_{\mathcal{X}}(\delta_\pi^i(x)-\delta_{\pi_n}^i(x))\left[(\delta_\pi^i(X)+\delta_{\pi_n}^i(X)) - 2\frac{I_x[(\nabla_i r(\theta))V_i(\theta)h_n^2(\theta)g(\theta)]}{I_x[V_i(\theta)h_n^2(\theta)g(\theta)]}\right]\\ &\quad\cdot I_x[V_i(\theta)h_n^2(\theta)g(\theta)]\mu(dx)\\ &= \int_{\mathcal{X}}[\delta_\pi^i(x)-\delta_{\pi_n}^i(x)]^2 I_x[V_i(\theta)h_n^2(\theta)g(\theta)]\mu(dx).\end{aligned} \tag{3.4}$$

Substituting (2.3) and (2.9) into (3.4) we have

$$\begin{aligned}\Delta_n^i &= \int_{\mathcal{X}} \left[\frac{I_x[\nabla_i(V_i(\theta)g(\theta))]}{(\eta+1)I_x[V_i(\theta)g(\theta)]} - \frac{I_x[\nabla_i(V_i(\theta)h_n^2(\theta)g(\theta))]}{(\eta+1)I_x[V_i(\theta)h_n^2(\theta)g(\theta)]}\right]^2 \\ &\quad \cdot I_X[V_i(\theta)h_n^2(\theta)g(\theta)]\mu(dX) \\ &\leq \frac{2}{(\eta+1)^2}\Big[\int_{\mathcal{X}} \left[\frac{I_x[\nabla_i(V_i(\theta)g(\theta))]}{I_x[V_i(\theta)g(\theta)]} - \frac{I_x[h_n^2(\theta)\nabla_i(V_i(\theta)g(\theta))]}{I_x[V_i(\theta)h_n^2(\theta)g(\theta)]}\right]^2 \\ &\quad \cdot \; I_x[V_i(\theta)h_n^2(\theta)g(\theta)]\mu(dx) \\ &\quad + \; 4\int_{\mathcal{X}} \left[\frac{I_x[V_i(\theta)h_n(\theta)g(\theta)\nabla_i h_n(\theta)]}{I_x[V_i(\theta)h_n^2(\theta)g(\theta)]}\right]^2 \cdot I_x[V_i(\theta)h_n^2(\theta)g(\theta)]\mu(dx)\,\Big] \\ &= \frac{2}{(\eta+1)^2}(B_n + 4A_n), \text{ say.} \end{aligned} \tag{3.5}$$

First, consider the term, A_n, in the right-hand side of (3.5). Now for each $x \in \mathcal{X}$, using Cauchy-Schwartz inequality,

$$\begin{aligned}&\{I_x[V_i(\theta)h_n(\theta)g(\theta)\nabla_i h_n(\theta)]\}^2 \\ \leq\; & I_x[V_i(\theta)g(\theta)h_n^2(\theta)] \cdot I_x[V_i(\theta)g(\theta)\{\nabla_i h_n(\theta)\}^2].\end{aligned} \tag{3.6}$$

Substituting (3.6) into A_n yields, by Fubini's theorem and condition (3.1) of Theorem 3.1,

$$\begin{aligned}A_n &\leq \int_{\mathcal{X}} I_x(V_i(\theta)g(\theta)[\nabla_i h_n(\theta)]^2)\mu(dx) \\ &= \int_{\Theta} [\nabla_i h_n(\theta)]^2 V_i(\theta)\pi(\theta)\, d\theta \to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.7}$$

Next, consider the term, B_n, in the right-hand side of (3.5). Using the Cauchy-Schwartz inequality and the fact that for all $n \geq 1$, $h_n^2(\theta) \leq M < \infty$ a.e. (Lebesgue measure) by (1) and (3) of Lemma 2.1, and then apply Fubini's theorem yield

$$B_n \leq M\int_{\mathcal{X}} I_x\left[V_i(\theta)g(\theta)[\frac{I_x[\nabla_i(V_i(\theta)g(\theta))]}{I_x[V_i(\theta)g(\theta)]} - \frac{\nabla_i(V_i(\theta)g(\theta))}{V_i(\theta)g(\theta)}]^2\right]\mu(dx).$$

$$\begin{aligned} &\leq M \int_{\mathcal{X}} I_x \left[\frac{[\nabla_i(V_i(\theta)g(\theta))]^2}{V_i(\theta)g(\theta)} \right] \mu(dx) \\ &= M \int_{\Theta} \frac{[\nabla_i(V_i(\theta)g(\theta))]^2}{V_i(\theta)g(\theta)} e^{\alpha\cdot\theta+\varphi(\theta)-(\eta+1)r(\theta)} d\theta. \end{aligned} \tag{3.8}$$

But,

$$\begin{aligned} \nabla_i(V_i(\theta)g(\theta)) &= \nabla_i[V_i(\theta)\pi(\theta)e^{-\alpha\cdot\theta-\varphi(\theta)+(\eta+1)r(\theta)}] \\ &= \{\nabla_i[V_i(\theta)\pi(\theta)] + V_i(\theta)\pi(\theta)[-\alpha_i - \nabla_i\varphi(\theta) + (\eta+1)\nabla_i r(\theta))]\} \\ &\cdot \ e^{-\alpha\cdot\theta-\varphi(\theta)+(\eta+1)r(\theta)}. \end{aligned} \tag{3.9}$$

Hence, (3.8) becomes, using the conditions (3.2) and (3.3) of Theorem 3.1 and (3.9),

$$\begin{aligned} B_n &\leq \int_{\Theta} \frac{\{\nabla_i[V_i(\theta)\pi(\theta)] + V_i(\theta)\pi(\theta)[-\alpha_i - \nabla_i\varphi(\theta) + (\eta+1)\nabla_i r(\theta)]\}^2}{V_i(\theta)\pi(\theta)} d\theta \\ &\leq 2\int_{\Theta} \left[\frac{[\nabla_i(V_i(\theta)\pi(\theta))]^2}{V_i(\theta)\pi(\theta)} + V_i(\theta)\pi(\theta)[-\alpha_i - \nabla_i\varphi(\theta) + (\eta+1)\nabla_i r(\theta))]^2 \right] d\theta \\ &< \infty. \end{aligned} \tag{3.10}$$

Recall that, from (3.5),

$$B_n = \int_{\mathcal{X}} b_n(X)\mu(dx) \quad \text{for all } n \geq 1 \tag{3.11}$$

where, for each $x \in \mathcal{X}$ and $n \geq 1$,

$$b_n = \left[\frac{I_x[\nabla_i(V_i(\theta)g(\theta))]}{I_x[V_i(\theta)g(\theta)]} - \frac{I_x[\nabla_i(V_i(\theta)g(\theta))h_n^2(\theta)]}{I_x[V_i(\theta)g(\theta)h_n^2(\theta)]} \right]^2 I_x[V_i(\theta)g(\theta)h_n^2(\theta)]. \tag{3.12}$$

Then using condition (3) of Lemma 2.1, (3.12) yields, for all $x \in \mathcal{X}$.

$$b_n(x) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Hence, by the Lebesgue's dominated convergence theorem, (3.10) and (3.13) gives

$$B_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Therefore, from (3.5), (3.7) and (3.14), we have $\Delta_n{}^i \to 0$ as $n \to \infty$ for all $i = 1, \cdots, p$. Thus, $\delta_\pi(X)$ is admissible for estimating $\nabla r(\theta)$ under the loss (2.1) by Lemma 2.1.

There are many choices of the sequence $\{h_n\}$ satisfying conditions (1), (2), and (3) of Lemma 2.1. When $\Theta = \mathrm{R}^p$, we use the sequence $\{h_n\}$ given in Brown and Hwang (1982) such that

$$h_1(\theta) = \begin{cases} 1, & |\theta| \le 1 \\ 0, & |\theta| > 1 \end{cases}$$

and

$$h_n(\theta) = \begin{cases} 1 & , \quad |\theta| \le 1 \\ 1 - \frac{\ln(|\theta|)}{\ln(n)} & , \quad 1 \le |\theta| \le n \\ 0 & , \quad |\theta| \ge n, \quad n = 2, 3, \cdots \end{cases} \tag{3.15}$$

and also, when $\Theta \ne \mathrm{R}^p$, we take the sequence $\{h_n\}$ given in Brown and Hwang (1982) such that

$$h_1(\theta) = \begin{cases} 1, & \Lambda(\theta) \le 1 \\ 0, & \Lambda(\theta) > 1 \end{cases}$$

and

$$h_n(\theta) = \begin{cases} 1 & , \quad \Lambda(\theta) \le 1 \\ 1 - \frac{\ln(\Lambda(\theta))}{\ln(n)} & , \quad 1 \le \Lambda(\theta) \le n \\ 0 & , \quad \Lambda(\theta) \ge n, \quad n = 2, 3, \cdots \end{cases} \tag{3.16}$$

where $\Lambda^2(\theta) = \sum_{i=1}^{p} \ln^2(|\theta_i|)$. Note that the sequences (3.15) and (3.16) satisfy the assumptions (1), (2), and (3) of Lemma 2.1. Hence, we have the following corollaries without proofs as applications of Theorem 3.1.

Corollary 3.1. Let $\Theta = \mathrm{R}^p$. If

$$\sum_{i=1}^{p} \int_S \frac{V_i(\theta)\pi(\theta)}{|\theta|^2 \ln^2(|\theta|)} d\theta < \infty, \tag{3.17}$$

where $S = \{\theta : |\theta| \geq 2\}$ and the conditions (2.2), (3.2), and (3.3) of Theorem 3.1 are satisfied, then $\delta_\pi(X)$ in (2.3) is admissible for estimating $\nabla r(\theta)$ under the loss (2.1).

Corollary 3.2. Let $\Theta \neq \mathrm{R}^p$. If

$$\sum_{i=1}^{p} \int_{S'} \frac{V_i(\theta)\pi(\theta)}{\theta_i^2 \Lambda^2(\theta) \ln^2(\Lambda(\theta))} d\theta < \infty, \tag{3.18}$$

where $S' = \{\theta : \Lambda(\theta) \geq 2\}$, and the conditions (2.2), (3.2), and (3.3) of Theorem 3.1 are satisfied, then $\delta_\pi(X)$ in (2.3) is admissible for estimating $\nabla r(\theta)$ under the loss (2.1).

4. EXAMPLES

In the following we use Corollary 3.1 for $\Theta = \mathrm{R}^p$ and Corollary 3.2 for $\Theta \neq \mathrm{R}^p$.

Example 4.1. Suppose that $X \sim N(\theta, I_p)$, $\theta \in \mathrm{R}^p$, I_p the $p \times p$ identity martrix. Then the density of X is of the form $f(x;\theta) = \exp\{\theta \cdot x - \frac{1}{2}\theta \cdot \theta\}$, with repect to $\sigma-$ finite measure $\mu(dx) = (2\pi)^{-p/2} \exp\{\frac{-1}{2}\sum_{i=1}^p x_i^2\}dx$, where $x \in \mathcal{X} = \mathrm{R}^p$ and $\theta \in \Theta = \mathrm{R}^p$. In this case $\varphi(\theta) = \frac{1}{2}\theta \cdot \theta$. It is desired to estimate $\nabla r(\theta) = \theta$ under a weighted quadratic loss

$$L(\theta, d) = \sum_{i=1}^{p} e^{-\theta_i^k}(d_i - \theta_i)^2, \tag{4.1}$$

where k is a non negative and even integer. Here $V_i(\theta) = e^{-\theta_i^k}$ and $r(\theta) = \frac{1}{2}\theta \cdot \theta$. Consider a prior density $\pi(\theta) = 1$. Then with $\alpha = 0$, $g(\theta) = \exp\{(\eta/2)\theta \cdot \theta\}$, $\eta \neq -1$. Now, we apply Corollary 3.1. A simple calculation shows that Condition (2.2) with $\alpha = 0$ is satisfied for $p \geq 1$ and $\eta \neq -1$. Now, we can easily check that Condition (3.3) with $\alpha = 0$ is satisfied for $p \geq 1$ if $\eta = 0$, and Condition (3.2) is also satisfied for $p \geq 1$ if $k = 0$. Finally, we consider condition (3.17). Then, transforming to p-dimensional spherical coordinates,

$$\sum_{i=1}^{p} \int_{\{|\theta| \geq 2\}} \frac{e^{-\theta_i^k}}{|\theta|^2 \ln^2 |\theta|} d\theta \quad \leq \quad p \int_{\{|\theta| \geq 2\}} \frac{1}{|\theta|^2 \ln^2 |\theta|} d\theta$$

$$\begin{aligned} &= p\beta_n \int_2^\infty \frac{r^{p-1}}{r^2 \ln^2 r} dr \\ &< \infty \quad \text{for } p = 1, 2, \end{aligned}$$

where $\beta_n = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{p-2} y_1 \sin^{p-3} y_2 \cdots \sin y_{p-2} \cdot dy_1 \, dy_2 \cdots dy_{p-1} < \infty$. Hence, $\delta_\pi(X) = X$ is admissible for $p = 1, 2$ under a weighted quadratic loss (4.1) with $k = 0$ by Corollary 3.1. Beak(1990) showed that for $p \geq 3$ $\delta(X) = \{1 - (p-2)/(\sum_{i=1}^p |X_i|^{2/b})^b\}X, b \geq 1$, dominate $\delta_\pi(X) = X$ under the sum of squared error loss and hence $\delta_\pi(X) = X$ is inadmissible for $p \geq 3$ under the sum of squared error loss . Note that $b = 1$ gives the ordinary James-Stein estimator.

Example 4.2. Let X be as in Example 4.1. It is desired to estimate $\nabla r(\theta) = \theta$ under a weighted quadratic loss

$$L(\theta, d) = \sum_{i=1}^p (1 + \theta_i^2)^{-1}(d_i - \theta_i)^2. \tag{4.2}$$

Here $V_i(\theta) = (1 + \theta_i^2)^{-1}$ and $r(\theta) = \frac{1}{2}\theta \cdot \theta$. Conder a prior density

$$\pi(\theta) = \prod_{i=1}^p (1 + \theta_i^2) \exp\{-(\eta/2)\theta \cdot \theta\}, \eta \neq -1.$$

Then with $\alpha = 0$, $g(\theta) = \prod_{i=1}^p (1 + \theta_i^2)$. Now, we apply Corollary 3.1. Condition (2.2) with $\alpha = 0$ is trivially satisfied for $p \geq 1$ and $\eta \neq -1$. The conditions (3.2) and (3.3) are equivalent in this case. And, it can be easily shown that Condition (3.3) with $\alpha = 0$ is satisfied for $p \geq 1$ if $\eta \geq 0$. Finally, a simple calculation shows that condition (3.17) is satisfied only for $p = 1$ and $\eta \geq 0$. Hence, $\delta_\pi(X) = X/(\eta + 1)$, $\eta \geq 0$, is admissible for $p = 1$ under a weighted quadratic loss (4.2). In particular, $\delta_\pi(X) = X$ is admissible for $p = 1$. Brown(1980) showed that the estimator $\delta_i(X) = X_i - c(sgn\ X_i)|X_i|^3/\{\sum |X_i|^4\}, i = 1, \cdots, p, 0 < c < 2[3p - 4]; p \geq 2$ dominates $\delta_\pi(X) = X$ for $p \geq 2$, and hence $\delta_\pi(X) = X$ is inadmissible for $p \geq 2$ under a weighted quadratic loss (4.2).

Example 4.3. Suppose that $X_i \sim$ Possion(λ_i), $\lambda_i > 0$, $i = 1, \cdots, p$ and X_i are independent. Then $X = (X_1, \cdots, X_p)$ have the density

$$f(x;\lambda)=\prod_{i=1}^{p}[e^{-\lambda_i}\lambda_i^{x_i}](x_i!)^{-1}, x_i=0,1,2\cdots,\lambda_i>0, i=1,\cdots,p$$

with respect to counting measure. Now we rewrite the density as the reparametrized form with $\ln\lambda_i=\theta_i$ for $i=1,\cdots,p$. Then, the density of X is of the form

$$f(x;\theta)=\exp(\theta\cdot x-\sum_{i=1}^{p}e^{\theta_i}), \theta\in\Theta=\mathsf{R}^p, x\in\mathcal{X}=\overset{p}{\times}_{i=1}\{0,1,\cdots\}$$

with respect to $\sigma-$finite measure $\mu(x)=(\prod_{i=1}^{p}x_i!)^{-1}\cdot$ (counting measure). In this case $\varphi(\theta)=\sum_{i=1}^{p}\exp\{\theta_i\}$. We want to estimate $\nabla_i r(\theta)=\exp\{\theta_i\}$ for $i=1,\cdots,p$ under a weighted quadratic loss

$$L(\theta,d)=\sum_{i=1}^{p}e^{-\theta_i}(d_i-e^{\theta_i})^2. \tag{4.3}$$

Here $V_i(\theta_i)=\exp\{-\theta_i\}$ and $r(\theta)=\sum_{i=1}^{p}e^{\theta_i}$. Consider a prior density

$$\pi(\theta)=\prod_{i=1}^{p}e^{\theta_i-\eta\cdot\exp\{\theta_i\}}, \eta\neq-1.$$

Then with $\alpha=0$, $g(\theta)=\prod_{i=1}^{p}e^{\theta_i}$, We apply Corollary 3.1. Condition (2.2) with $\alpha=0$ is trivially satisfied for $p\geq1$ and $\eta\neq-1$. It is enough to check Condition (3.3) because the Conditions (3.2) and (3.3) are equivalent with $\alpha=0$. Then we can easily show that Condition (3.3) is satisfied either for $p=1$ if $\eta\geq0$, or for $p\geq2$ if $\eta=0$, and Condition (3.17) holds for $p=1$ if $\eta\geq0$. Thus $\delta_\pi(X)=X/(1+\eta)$, $\eta\geq0$ is admissible for $p=1$ under a weighted quadratic loss (4.3). In particular, $\delta_\pi(X)=X$ is admissible for $p=1$. Clevenson and Zidek (1975) showed that for $p\geq2$, $\delta(X)=[1-(p-1)/\{\sum_{i=1}^{p}X_i+p-1\}]X$ dominates $\delta_\pi(X)=X$, and hence $\delta_\pi(X)=X$ is inadmissible for $p\geq2$ under a weighted quadratic loss (4.3).

Example 4.4. Let $X=(X_1,\cdots,X_p)$ have the density

$$f(x;\lambda) = \frac{\prod_{i=1}^{p} \lambda_i^s \cdot x_i^{s-1}}{\Gamma^p(s)} e^{-\lambda_i \cdot X_i}, x \in \mathcal{X} = \mathsf{X}_{i=1}^{p}(0,\infty)$$

$\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathsf{X}_{i=1}^{p}(0,\infty)$ with respect to Lebesque measure. Now we reparametrize $-\lambda_i = \theta_i$, for $i = 1, \cdots, p$. Then X have the density of the of the form

$$f(x;\theta) = e^{\theta \cdot x + s \sum_{i=1}^{p} \ln(-\theta_i)}$$

with respect to σ-finite measure $\mu(dx) = [\prod_{i=1}^{p} x_i^{s-1}/\{\Gamma^p(s)\}]\, dx$ where $\theta \in \Theta = \mathsf{X}_{i=1}^{p}(-\infty,0), x \in \mathcal{X} = \mathsf{X}_{i=1}^{p}(0,\infty)$. Then $\varphi(\theta) = -s \sum_{i=1}^{p} \ln|\theta_i|$ and we want to estimate $\nabla_i r(\theta) = 1/|\theta_i|$, $i = 1, \cdots, p$ under a weighted quadratic loss

$$L(\theta, d) = \sum_{i=1}^{p} |\theta_i|^{m+2}(d_i - 1/|\theta_i|)^2, m \text{ is an integer.} \tag{4.4}$$

Here, $V_i(\theta) = |\theta_i|^{m+2}$ and $r(\theta) = \sum_{i=1}^{p} \ln(-\theta_i)$. Fix $\eta \neq -1$, $\alpha \in \mathsf{R}^p$ and consider a prior distribution $\prod$ with the density $\pi(\theta) = \prod_{i=1}^{p} |\theta_i|^{\eta-s-m-1}$. Then with $\alpha = 0$, $g(\theta) = \prod_{i=1}^{p} |\theta_i|^{-m-2}$. We apply Corollary 3.2. Condition (2.2) with $\alpha = 0$ is trivially satisfied. Simple algebra shows that Condition (3.3) with $\alpha = 0$ is satisfied for $p \geq 1$ if $\eta = s - 1$. Finally, Condition (3.18) is satisfied if

$$\sum_{i=1}^{p} \int_{\{\theta \in \Theta : \Lambda(\theta) \geq 2\}} \frac{|\theta_i|^{\eta-s-1} \prod_{j\neq i}^{p} |\theta_j|^{\eta-m-s-1}}{\Lambda^2(\theta) \ln^2(\Lambda(\theta))} d\theta < \infty.$$

A lengthy calculation by transforming to p-dimensional spherical coordinates shows that this requires either $p = 1$ if $\eta = s$ or $p = 2$ if $\eta = s$ and $m = 0$. Therfore Corollary 3.2 fails. In this situation Dong(1990) showed by using different sufficient conditions that $\delta_\pi(X) = X/(s+1)$ is admissible either for $p = 1$ or for $p = 2$ and $m = 0$ under the loss (4.4). Also, Berger(1980) showed that for $p \geq 3, m = 0$, $\{X_i/(1+s)\}[1 - \{c(1+s)\ln X_i\}/\{b + \sum s(s+1)(\ln X_i)^2\}]$ dominates $\delta_\pi(X) = X/(s+1)$ where $s \geq 3$, $b > 0$, and

$$0 < c \leq \frac{2\{p-2-\sum(s+1)^{-1} - 3b^{-1/2}[\sum s/(s+1)]^{1/2}\}}{1+(2p/b)+(16/[27bp]+b^{-1/2}[4p+\sum\{s(s+1)\}^{-1}]^{1/2}}.$$

On the other hand, Das Gupta (1986) showed that for $p \geq 2, m \neq 0, \{X_i/(1+s)\} + X_i^{1+m/2} \cdot (\prod_{i=1}^{p} X_i^{-(m/2p)})$ dominates $\delta_\pi(X) = X/(1+s)$.

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