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A Note on Eigen Transformation of a Correlation-type Random Matrix

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ABSTRACT

It is well known that distribution of functions of eigen values and vectors of a certain matrix plays an important role in multivariate analysis. This paper deals with the transformation of a correlation-type random matrix to its eigen values and vectors. Properties of the transformation are also considered. The results obtained are applied to express the joint distribution of eigen values and vectors of the correlation matrix when sample is taken from a m-variate spherical distribution.

KEYWORDS: Eigen transformation, Exterior product, Spherical distribution, Correlation-type random matrix, James' approach to Jacobian.

1. INTRODUCTION

Distribution problems in multivariate analysis are often related to those of eigen values and/or vectors of a certain random matrix. A variety of exact distributional results on the eigen system are easily found when dealing with Wishart(or sample covariance) matrix. However sample correlation matrix produces no such results, mainly due to the difficulty in deriving Jacobian of the corresponding eigen transformation (see Scheunemeyer & Lucantoni,1978). Specifically the complication comes from the fact that correlation matrix has 1 in the diagonal, causing some constraints

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on its associated eigen values and vectors. In this paper, we define a correlation-type random matrix, consider characteristics of its eigen tranformation, and finally obtain the Jacobian of it using James' approach(James,1954). The results will be useful to obtain the joint distribution of eigen values and vectors of the correlation-type random matrix.

2. EIGEN TRANSFORMATION OF THE CORRELATION-TYPE RANDOM MATRIX

A correlation-type random matrix is defined as follow.

Definition 1. A real $m \times m$ symmetric nonsingular matrix is a correlation-type random matrix, denoted by $R = (r_{ij})$, if it has fixed constants in the diagonal and random elements in the off-diagonal. That is,

for $i, j = 1, 2, \cdots, m$,

$$r_{ij} = \begin{cases} c_i & \text{if } i = j, \text{ constant } c_i \\ r_{ji} & \text{if } i \neq j. \end{cases} \tag{2.1}$$

Thus the usual correlation or similarity matrix would be considered as a typical form of the R.

2.1 Characteristics of the eigen-transformation of R

Consider the following spectral decomposition of R

$$R = HLH' \tag{2.2}$$

where $L = diag(l_1, l_2, \cdots, l_m)$ is a diagonal matrix of the eigen values of R with l_j as the j^{th} diagonal element, and $H = (h_{ij}) = (\mathbf{h}_1 \ \mathbf{h}_2 \ \cdots \ \mathbf{h}_m)$ is an orthogonal matrix of the eigen vectors $\mathbf{h}_j$ associated with l_j, $j = 1, 2, \cdots, m$. Now we can let, without loss of generality, that $l_1 < l_2 < \cdots < l_m$ be the ordered eigen values of R since the probability that any eigen values of R are equal is zero. Note that the sum of all eigen values of R is set to a fixed value, i.e. $\sum_{j=1}^{m} l_j = \sum_{j=1}^{m} c_j$.

To explain some dimensional properties of this eigen transformation, let's define the following spaces. For R, L and H given in (2.1) and (2.2),

$$
\begin{aligned}
\mathbf{S}_m(R \mid C) &= \{R \mid R \text{ has common constants } c_1, c_2, \cdots, c_m \text{ in the diagonal}\} \\
\mathbf{S}_m(R \mid L, C) &= \{R \mid R \in S_m(R|C), \text{ and } R \text{ has common eigen values } l_i\text{'s}, \\
&\quad i = 1, \cdots, m\} \\
\mathbf{S}_m(L \mid C) &= \{L \mid \textstyle\sum_{j=1}^m l_j = \sum_{j=1}^m c_j \text{ for given constants } c_1, c_2, \cdots, c_m\} \\
\mathbf{S}_m(H \mid L, C) &= \{H \in O(m), \text{ the orthogonal group of order } m \mid \textstyle\sum_{j=1}^m l_j h_{ij}^2 = c_i \\
&\quad \text{for given } l_j\text{s and } c_i\text{s}, \ i, j = 1, 2, \cdots, m\}
\end{aligned} \tag{2.3}
$$

Then the random elements of those matrices $R \in \mathbf{S}_m(R \mid C)$, $R \in \mathbf{S}_m(R \mid L, C)$, $L \in \mathbf{S}_m(L \mid C)$, and the constrained manifold $H \in \mathbf{S}_m(H \mid L, C)$ can be regarded as the coordinates of a point on a $m(m-1)/2$, $(m-1)(m-2)/2$, $(m-1)$, and $(m-1)(m-2)/2$ dimensional surface in Euclidean m^2-space, respectively.

Furthermore we note the following characteristics of the transformation. If L is fixed, the transformation in (2.2) is equivalent to the transformation of $\mathbf{S}_m(R \mid L, C) \to \mathbf{S}_m(H \mid L, C)$. And, for $R \in \mathbf{S}_m(R \mid C)$ the transformation in (2.2) can be expressed as the union of $\mathbf{S}_m(H|L, C)$ over all partitions $L \in \mathbf{S}_m(L|C)$. That is, the transformation becomes $\mathbf{S}_m(R \mid C) \to \mathbf{S}_m(H, L \mid C) \equiv \bigcup_{L \in \mathbf{S}_m(L|C)} \mathbf{S}_m(H \mid L, C)$.

This transformation is not, however, one-to-one correspondence since R determines 2^m matrices of H such as $H = (\pm \mathbf{h}_1 \ \pm \mathbf{h}_2 \ \cdots \ \pm \mathbf{h}_m)$ which satisfies (2.2) unless we impose some constraints, e.g. the first element in each column of H be non-negative. This restricts the range of H(as R varies) to a $2^{-m}th$ part of $\mathbf{S}_m(H \mid L, C)$. Hence when we make the transformation R to (L, H), and integrate it with respect to the total differential over $H \in \mathbf{S}_m(H \mid L, C)$, the result should be divided by 2^m.

2.2 Exterior products of R and (L, H)

Partition L and H as follows:

$$
H = (H_1 \ \mathbf{h}_m) \qquad L = \begin{pmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & l_m \end{pmatrix} \tag{2.4}
$$

where H_1, a submatrix of H, consists of the first $(m-1)$ columns of H, and L_1 is the first $(m-1) \times (m-1)$ subdiagonal matrix of L. Then we have

$$
\begin{aligned}
R &= HLH' \\
&= H_1 L_1 H_1' + l_m \ \mathbf{h}_m \ \mathbf{h}_m'
\end{aligned} \tag{2.5}
$$

If we differentiate both sides of (2.5), premultiply H_1', and postmultiply H_1 on dR, we have

$$
H_1'(dR)H_1 = H_1'(dH_1)L_1 + (dL_1) - L_1 H_1'(dH_1) \tag{2.6}
$$

since $H_1'(dH_1) = -(dH_1')H_1$, a skew-symmetric matrix.

Since the typical $(i, j)^{th}$element of the RHS of (2.6) is

$$\begin{cases} dl_i & \text{for } i = j, \\ (l_j - l_i)\mathbf{h}_i'(d\mathbf{h}_j) & \text{for } i \neq j, \quad i,j = 1,2,\cdots,m-1, \end{cases} \tag{2.7}$$

the corresponding exterior product is simply given by

$$< H_1'(dH_1)L_1 + (dL_1) - L_1 H_1'(dH_1) > = \prod_{j>i}^{m-1} (l_j - l_i) \cdot \bigwedge_{i=1}^{m-1} (dl_i) \cdot \bigwedge_{j>i}^{m-1} \mathbf{h}_i'(d\mathbf{h}_j) \tag{2.8}$$

where the symbol $< dX >$ denotes the exterior product of the distinct elements of dX, and $\bigwedge$ denotes the corresponding "wedge" product.

On the other hand, the exterior product of the LHS of (2.6) is rather complicate. First, it is easy to see that the $(i,j)^{th}$ element of $H_1'(dR)H_1$ can be written as

$$(H_1'(dR)H_1)_{ij} = \sum_{p>q}^{m} {}_{pq}\mathbf{h}_{ij}(dr_{pq}), \qquad i,j = 1,2,\cdots,(m-1) \tag{2.9}$$

where ${}_{pq}\mathbf{h}_{ij} = h_{pi}h_{qj} + h_{pj}h_{qi}$ for $m-1 \geq i \geq j \geq 1,\ m \geq p > q \geq 1.$

Hence the exterior product of symmetric $H_1'(dR)H_1$ becomes

$$< H_1'(dR)H_1 > = \bigwedge_{i \geq j}^{m-1} \sum_{p>q}^{m} {}_{pq}\mathbf{h}_{ij}(dr_{pq}) \tag{2.10}$$

Now for $p = 2,3,\cdots,m$, $i = 1,2,\cdots,m-1$, let $\psi_{p-1,i}$ be a matrix of size $(p-1)\times i$ with ${}_{pq}\mathbf{h}_{ij}$ as the $(q,j)^{th}$ element, and let $\psi_{p-1,i}$ itself be the $(p-1,i)^{th}$ submatrix of a matrix Ψ, which is of size $(m(m-1)/2)\times(m(m-1)/2)$. That is,

$$\psi_{p-1,i} = \begin{pmatrix} {}_{p1}\mathbf{h}_{i1} & {}_{p1}\mathbf{h}_{i2} & \cdots & {}_{p1}\mathbf{h}_{ij} & \cdots & {}_{p1}\mathbf{h}_{ii} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ {}_{pq}\mathbf{h}_{i1} & {}_{pq}\mathbf{h}_{i2} & \cdots & {}_{pq}\mathbf{h}_{ij} & \cdots & {}_{pq}\mathbf{h}_{ii} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ {}_{p,p-1}\mathbf{h}_{i1} & {}_{p,p-1}\mathbf{h}_{i2} & \cdots & {}_{p,p-1}\mathbf{h}_{ij} & \cdots & {}_{p,p-1}\mathbf{h}_{ii} \end{pmatrix} \tag{2.11}$$

and

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1i} & \cdots & \psi_{1,m-1} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \psi_{p-1,1} & \psi_{p-1,2} & \cdots & \psi_{p-1,i} & \cdots & \psi_{p-1,m-1} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \psi_{m-1,1} & \psi_{m-1,2} & \cdots & \psi_{m-1,i} & \cdots & \psi_{m-1,m-1} \end{pmatrix}. \tag{2.12}$$

Let $\mathbf{dr} = (dr_{21} : dr_{31}\ dr_{32} : \cdots : dr_{i1}\ dr_{i2} \cdots dr_{i,i-1} : \cdots : dr_{m1}\ dr_{m2} \cdots dr_{m,m-1})'$, a $(m(m-1)/2) \times 1$ column vector of differentials. It is actually a vector of rolled-out lower half elements of dR . Then we have the following two lemmas.

Lemma 1. The exterior product of $H_1'(dR)H_1$

$$< H_1'(dR)H_1 > = < \Psi'(d\mathbf{r}) > = \det(\Psi) \bigwedge_{p>q}^{m} (dr_{pq}) \tag{2.13}$$

Proof. By Farrel's(1985) lemma 6.2.10.

Lemma 2.

$$\det(\Psi) = 2^{m-1} \cdot \prod_{i=1}^{m} \det(H_1(-i)) \tag{2.14}$$

where $H_1(-i)$ is a matrix of size $(m-1) \times (m-1)$ formed from H_1 with its i^{th} row deleted.

Proof. By mathematical inductions.

By combining the above two lemmas, the exterior product of $H_1'(dR)H_1$ becomes

$$< H_1'(dR)H_1 > = 2^{m-1} \cdot \prod_{i=1}^{m} \det(H_1(-i)) \bigwedge_{p>q}^{m} (dr_{pq}). \tag{2.15}$$

As a consequence of (2.8) and (2.15), we have the following theorem which provides a relationship between exterior products on both sides of (2.6).

Theorem 3.

$$\bigwedge_{p>q}^{m} (dr_{pq}) = \frac{\prod_{j>i}^{m-1}(l_j - l_i)}{2^{m-1}\prod_{i=1}^{m}\det(H_1(-i))} \bigwedge_{i=1}^{m-1} (dl_i) \bigwedge_{j>i}^{m-1} \mathbf{h}_i'(d\mathbf{h}_j) \tag{2.16}$$

This result obviously plays a central role in obtaining the Jacobian of the eigen transformation (2.2) of R.

3. APPLICATIONS

As an application we have the following expression for the joint distribution of (L, H) from R. Let $f(R)dR$ be the density of R. Then the joint density of (L, H) is given by, directly from (2.16),

$$\frac{\prod_{j>i}^{m-1}(l_j - l_i)}{2^{m-1}\prod_{i=1}^{m}\det(H_1(-i))}\cdot(f(R)|_{R=HLH'})\bigwedge_{i=1}^{m-1}(dl_i)\bigwedge_{j>i}^{m-1}\mathbf{h}_i'(d\mathbf{h}_j) \qquad (3.1)$$

where $f(R)|_{R=HLH'}$ stands for the density function of R with R being replaced by HLH'.

Suppose N-dimensional m random vectors, $Y_1, Y_2, \cdots, Y_m$ are all independent where Y_i has an N-variate spherical distribution with $P(Y_i = 0) = 0$ for all $i = 1, 2, \cdots, m$. Then the density function of the sample correlation matrix R of Y_i's (Murhead, 1982) is given by

$$c(n,m)[\det(R)]^{(n-m-1)/2}\bigwedge_{p>q}^{m}(dr_{pq}) \qquad (3.2)$$

where $n = N-1$, multivariate gamma function $\Gamma_m(a) = \pi^{m(m-1)/4}\cdot\prod_{i=1}^{m}\Gamma(a-(i-1)/2)$, and $c(n,m) = [\Gamma(n/2)]^m/\Gamma_m(n/2)$. For this special case, the joint distribution of (H, L) can be expressed as

$$\frac{c(n,m)}{2^{(}m-1)}\{[\prod_{i=1}^{m-1} l_i(m-\sum_{j=1}^{m-1} l_j)]^{(n-m+1)/2}\cdot\prod_{j>i}^{m-1}(l_j-l_i)\bigwedge_{i=1}^{m-1} dl_i\}$$
$$\cdot\{\frac{1}{\prod_{i=1}^{m}\det(H_1(-i))}\bigwedge_{j>i}^{m-1}\mathbf{h}_i'(d\mathbf{h}_j)\} \qquad (3.3)$$

REMARKS

(1) It is easy to show that $\det(H_1(-i)) = (-1)^{i+1}h_{im}$. Hence $\prod_{i=1}^{m}\det(H_1(-i))$ in (3.3) can be written as the product of all elements in the last column of H.

(2) We need to intergrate out terms associated with H_1 over the mainfold $\mathbf{S}_m(H \mid L, C)$ in (3.3) if it is desired to obtain a marginal distribution of the eigenvalues, L. However the intergration

$$\int_{\mathbf{S}_m(H|L,C)}\frac{1}{\prod_{i=1}^{m}h_{mi}}\bigwedge_{j>i}^{m-1}\mathbf{h}_i'(d\mathbf{h}_j)$$

seems to be increasingly difficult simply because of the complicated integrating region in $\mathbf{S}_m(H \mid L, C)$. No futher development, unfortunately, is provided.

(3) When sample is taken from a bivariate normal distribution ($m = 2$), the null density function of l_1 for a correlation matrix under the hypothesis of internal independence is given by using (3.3)

$$\frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)}[l_1(2-l_1)]^{(n-3)/2}dl_1$$

Here we do not need to consider the integration over $\mathbf{S}_m(H \mid L, C)$ since the corresponding L and H are, for this special case of $m = 2$, as follows.

$$L = \begin{pmatrix} l_1 & 0 \\ 0 & 2-l_1 \end{pmatrix}$$
$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2}$$
$$H_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$$

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