Journal of the Korean
Statistical Society
Vol. 22, No. 2, 1993

A Note on the Invariance Principle for Associated Sequences

Tae-sung Kim[1] and Kwang-Hee Han[2]

ABSTRACT

In this note we consider other type of tightness than that of Birkel(1988) and prove an invariance principle for nonstationary associated processes by an application of the central limit theorem of Cox and Grimmett(1984),thus avoiding the argument of uniform integrability. This result is an extension to the nonstationary case of an invariance principle of Newman and Wright(1981) as well as an improvement of the central limit theorem of Cox and Grimmett (1984) .

AMS 1980 Subject Classifications: Primary 60B10; Secondary 60F05.
KEYWORDS: tightness, association, central limit theorem, invariance principle.

1. INTRODUCTION AND NOTATION

Throughout this paper let $\{X_j : j \in N\}$ be a sequence of random variables on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $EX_j = 0, EX_j^2 < \infty$. For $n \in N$, put

$$S_n = \sum_{j=1}^{n} X_j \text{ and } \sigma_n^2 = ES_n^2.$$

[1] Department of Statistics, Won-Kwang University, Iri, 540-749, Korea
[2] Department of Computer Science, Chun-Buk San Up University, Kunsan, 573-400, Korea

$\{X_j : j \in N\}$ is said to satisfy the central limit theorem, if $\sigma_n^{-1} S_n \to N(0,1)$ weakly. Define random elements W_n in $\mathcal{D}[0,1]$ endowed with the Skorokohod topology(see [1], §14) by

$$W_n(t) = \sigma_n^{-1} S_{[nt]} \ , \ t \in [0,1] \ , \ n \in N, \tag{1.1}$$

and let W denote a random element in $\mathcal{D}[0,1]$ with Wiener measure as its distribution. If the distribution of W_n converges weakly to Wiener measure we write $W_n \xrightarrow{\mathcal{D}} W$ and say that $\{X_j : j \in N\}$ fulfills the invariance principle(i.p).

In this paper we consider the invariance principle for sequences satisfying association. A finite collection $\{X_1, X_2, ..., X_m\}$ of random variables is associated if for any two coordinatewise nondecreasing functions f_1,f_2 on R^m such that $\widehat{f_i}$=$f_i(X_1,,,X_m)$ has variance for i=1,2, there holds $Cov(\widehat{f_1}, \widehat{f_2}) \geq 0$. An infinite collection is associated if every finite subcollection is associated (cf. Esary, Proschan and Walkup[5]). Associated processes are of considerable use in physics and statistics and have been investigated in recent years to a great extent (see, for example, Newman [6] and the references therein). Newman[6] has shown that associated processes satisfy the central limit theorem. Our aims of this paper are to provide other type of tightness than that of Birkel[2] and to improve a central limit theorem of Cox and Grimmett [4] to an invariance principle by adding the condition that $(\sigma_n^2/n) \to \sigma^2 \in (0, \infty)$.

Our result is also an extension to the nonstationary case of an invariance principle of Newman and Wright[7]. In section 2, We obtain a maximal inequality and a probability inequality (Theorems 2.2 and 2.3)which are generalizations of those of Newman and Wright[7] and use them in the proof of our tightness. Newman and Wright[8] derived martingale type inequalities related to Theorem 2.3 and Birkel[2] used them to obtain tightness for nonstationary associated processes. In section 3, we have another proof of the tightness of $\{W_n\}$ and prove the in variance principle for a nonstationary associated processes by an application of the central limit theorem of Cox and Grimmet[4], thus avoiding an argument for the uniform integrability.

2. PRELIMINARIES

Cox and Grimmet[4] weakened the assumption of stationarity and replaced it by certain conditions on the moments of random variables. Using the coefficient

$$u(n) = \sup_{k \in N} \sum_{j:|j-k| \leq n} Cov(X_j, X_k), n \in N \cup \{0\},$$

they obtained the following central limit theorem:

Theorem 2.1. (Cox,Grimmet(1984)). Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0, EX_j^2 = 0 < \infty$. Assume

$$u(n) \longrightarrow_n 0, \quad u(0) < \infty, \tag{2.1}$$
$$\inf_{j \in N} Var(X_j) > 0, \tag{2.2}$$
$$\sup_{j \in N} E|X_j|^3 < \infty. \tag{2.3}$$

Then $\{X_j : j \in N\}$ satisfies the central limit theorem, that is, $\sigma_n^{-1} S_n$ is asymptotically normally distributed.

In order to use Theorem 1 of Newman and Wright(1981) to the nonstationary case a slight variation is made in the following lemma and this may be considered as a generalization of that of [8].

Lemma 2.2. Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0, EX_j^2 < \infty$. Define for $n \in N$, $m \in N \cup \{0\}$,

$$S_{m,n} = S_{n+m} - S_m$$

and

$$M_{m,n} = max(S_{m,1}, S_{m,2}, ..., S_{m,n}).$$

Then

$$E(M_{m,n}^2) \leq Var(S_{m,n}). \tag{2.4}$$

Proof. Define

$$K_{m,n} = min(X_{2+m} + \cdots + X_{n+m}, X_{3+m} + \cdots + X_{n+m}, \cdots, X_{n+m}, 0),$$
$$L_{m,n} = max(X_{2+m}, X_{2+m} + X_{3+m}, \cdots, X_{2+m} + \cdots + X_{n+m}),$$

and $J_{m,n} = max(0, L_{m,n})$. Then we obtain that $Cov(X_{1+m}, K_{m,n}) \geq 0$, since $K_{m,n} = X_{2+m} + \cdots + X_{n+m} - J_{m,n}$ is a nondecreasing function of the X_i's , that $J_{m,n}^2 \leq L_{m,n}^2$ pointwise and that $M_{m,n} = X_{1+m} + J_{m,n}$ and thus

$$\begin{aligned} E(M_{m,n}^2) &= E(X_{1+m} + J_{m,n})^2 \\ &= Var(X_{1+m}) + 2Cov(X_{1+m}, J_{m,n}) + E(J_{m,n}^2) \\ &= Var(X_{1+m}) + 2Cov(X_{1+m}, X_{2+m} + \cdots + X_{n+m}) \\ &\quad -2Cov(X_{1+m}, K_{m,n}) + E(J_{m,n}^2) \\ &\leq Var(X_{1+m}) + 2Cov(X_{1+m}, X_{2+m} + \cdots + X_{n+m}) + E(L_{m,n}^2). \end{aligned} \tag{2.5}$$

The proof is completed by induction on m since $E(L_{m,n}^2) \le Var(X_{2+m} + \cdots + X_{n+m})$ which together with (2.5) yields (2.4).

Remark.
(A) If we put $m = 0$,then the result of this lemma automatically implies that of Newman and Wright(1981).
(B) A slight variation of the above proof shows that (2.4) remains valid with $M_{m,n}$ replaced by $S_{m,n}^{(j)}$, the j th order statistic of $(S_{m,1}, S_{m,2}, \cdots, S_{m,n})$.

We next define for $n \in N, m \in N \cup \{0\}$

$$S_{m,n}^* = max(0, S_{m,1}, S_{m,2}, ..., S_{m,n}), \quad s_{m,n}^2 = ES_{m,n}^2,$$

where $S_{m,n} = S_{n+m} - S_m$. The following inequalities are extensions of those in Newman and Wright[8] to the nonstationary case and will be used to provide the tightness needed for our invariance principle in the nonstationary case;

Theorem 2.3. For $\lambda_2 > \lambda_1 > 0$,

$$P(S_{m,n}^* \ge \lambda_2) \le (1 - s_{m,n}^2/(\lambda_2 - \lambda_1)^2)^{-1} P(S_{m,n} \ge \lambda_1), \tag{2.6}$$
$$P(max(|S_{m,1}|, ..., |S_{m,n}|) \ge \lambda s_{m,n}) \le 2P(|S_{m,n}| \ge (\lambda - \sqrt{2})s_{m,n}). \tag{2.7}$$

Proof. For $0 \le \lambda_1 < \lambda_2$,

$$\begin{aligned} P(S_{m,n}^* \ge \lambda_2) &\le P(S_{m,n} \ge \lambda_1) + P(S_{m,n-1}^* \ge \lambda_2, S_{m,n-1}^* - S_{m,n} \ge \lambda_2 - \lambda_1) \\ &\le P(S_{m,n} \ge \lambda_1) + P(S_{m,n-1}^* \ge \lambda_2) P(S_{m,n-1}^* - S_{m,n} > \lambda_2 - \lambda_1) \\ &\le P(S_{m,n} \ge \lambda_1) + P(S_{m,n}^* \ge \lambda_2) E((S_{m,n-1}^* - S_{m,n})^2)/(\lambda_2 - \lambda_1)^2. \end{aligned} \tag{2.8}$$

Now Theorem 2.2 with X_{i+m} replaced by $Y_{i+m} = -X_{n-i+1+m}$ yields that

$$\begin{aligned} E([S_{m,n-1}^* - S_{m,n}]^2) &= E([max(Y_{1+m}, Y_{1+m} + Y_{2+m}, \cdots, Y_{1+m} + Y_{2+m}, \cdots + Y_{n+m})]^2) \\ &\le E(S_{m,n}^2) = s_{m,n}^2, \end{aligned} \tag{2.9}$$

which together with (2.8) yields (2.6) for $(\lambda_2 - \lambda_1)^2 \ge s_{m,n}^2$. By adding to (2.6) the analogous inequality with each X_{i+m} replaced by $-X_{i+m}$ in (2.6), and by choosing $\lambda_2 = \lambda s_{m,n}$, $\lambda_1 = (\lambda - \sqrt{2})s_{m,n}$, (2.7) will be obtained.

Remark.
(A) The second inequality in (2.8) follows from the fact that $S_{m,n-1}^*$ and $S_{m,n} - S_{m,n}^*$ are associated since they are both non-decreasing functions of the $X_{i+m}'s$ and the fact that $P(X > x, Y > y) \ge P(X > x)P(Y > y)$ for associated random variables X and Y.

(B) (2.7) will yield by the standard argument the needed tightness of the distributions of $W_n^{\prime}s$ to obtain the desired convergence in distibution (see the proof of Theorem 8.4 in Billingsley[1]).
(C) If we put $m = 0$ the result of this theorem automatically imply those of Newman and Wright(1981).

3. AN INVARIANCE PRINCIPLE

Theorem 3.1. Let $\{X_j : j \in N\}$ be a sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume that $\{X_j : j \in N\}$ satisfies (2.1), (2.2) and (2.3). If $\{X_j; j \in N\}$ satisfies

$$n^{-1}\sigma_n^2 \longrightarrow_n \sigma^2 \in (0, \infty). \tag{3.1}$$

Then $\{X_j\}$ fulfills the invariance principle.

Proof. Certainly since (2.1),(2.2) and (2.3) are fulfilled $\{X_j\}$ satisfies the central limit theorem (see Theorem 2.1). We first show that the finite-dimensional distributions of the W_n converge to those of W. It follows from (3.1) and the fact that $[nt]/n \longrightarrow_n t$

$$\sigma_n^{-2}\sigma_{[nt]}^2 \longrightarrow_n t \quad , for \quad t > 0 \tag{3.2}$$

and hence (3.2) and Theorem 2.1(central limit theorem) yield

$$\sigma_n^{-1}S_{[nt]} \xrightarrow{\mathcal{D}} N(0, t). \tag{3.3}$$

By a simple consequence of the estimate

$$0 \le \sigma_n^{-2}E(S_{nj} - S_{ni})(S_{nl} - S_{nk}) \le \sigma_n^{-2} \sum_{r=1}^{n(j-i)} u(r),$$

for $i \le j \le k \le l \in N \cup \{0\}$ and assumptions (2.1) and (3.1), we have

$$\sigma_n^{-2}E((S_{nj} - S_{ni})(S_{nl} - S_{nk})) \longrightarrow_n 0 \;\; for \;\; i \le j \le k \le l \in N \cup \{0\}. \tag{3.4}$$

and (3.2), (3.4),and Lemma 2 of Birkel[2] yield

$$Cov(U_{n,i}, U_{n,j}) \longrightarrow_n 0 \quad \text{for all} \quad i(\neq)j. \tag{3.5}$$

where $U_{n,i} = W_n(t_i) - W_i(t_{i-1})$, $0 \le t_1 \le \cdots \le t_k \le 1$. Thus if $(U_1, ..., U_k)$ is a limit in distrbution of any subsequence of $(U_{n,1}, ...U_{n,k})$, the $U_i^{\prime}s$ would be associated and uncorrelated random variables and hence independent by Corollary 3 of Newman[7](or the remark following Theorem 1 of Newman and Wright[8]). This yields

$$U_{n,i} = W_n(t_i) - W_n(t_{i-1}) \longrightarrow N(0, t_i - t_{i-1}), \tag{3.6}$$

according to (3.3) and this, together with (3.6), shows that the finite dimensional distributions of W_n converge to those of the standard Wiener process. Next applying (2.7) in Theorem 2.3 and writing $\lambda' = \lambda(\sigma_n / s_{m,n})$, we have for $\lambda' \geq 2\sqrt{2}$,

$$\begin{aligned} P\{max_{i \leq n}|S_{i+m} - S_m| \geq \lambda\sigma_n\} &= P\{max_{i \leq n}|S_{i+m} - S_m| \geq \lambda'^{s}_{m,n}\} \\ &\leq 2P\{|S_{n+m} - S_m| \geq \frac{1}{2}\lambda'^{s}_{m,n}\}. \end{aligned}$$

By the central limit theorem and Chebyshev's inequality,

$$\begin{aligned} P\{|S_{n+m} - S_m| \geq \frac{1}{2}\lambda' s_{m,n}\} &\longrightarrow_n P\{|N| \geq \frac{1}{2}\lambda'\} \\ &\leq \left(\frac{8}{\lambda'^3}\right) E\{|N|^3|\} \\ &= \frac{8}{\lambda^3}(s_{m,n}/\sigma_n)^3 E\{|N|^3\}. \end{aligned}$$

Since

$$\begin{aligned} 0 &\leq \sigma_n^{-2} s_{m,n}^2 = \sigma_n^{-2} E(S_{m+n} - S_m)^2 \\ &\leq \sigma_n^{-2}(ES_{n+m}^2 - ES_m^2) \longrightarrow_n 1 \end{aligned}$$

if ϵ is positive, we have

$$\lim_{n \to \infty} \sup P\{max_{i \leq n}|S_{i+m} - S_m| \geq \lambda\sigma_n\} \leq \epsilon/\lambda^2$$

for sufficiently large λ. Therefore tighteness now follows by Theorem 8.4 of Billingsley(1968).

Remark.
(A) For a strictly stationary sequence of random variables condition(7) in Theorem 3 of Newman and Wright[8] implies $u(0) = \sigma^2$, $u(n) = 2\sum_{j=n+1}^{\infty} Cov(X_1, X_j)$, $n \in N$, and hence (2.1) and (2.2) are automatically satisfied and condition (7) in Theorem 3 of [8] obviously implies (3.1) (see [1] ,§20 ,Lemma 3). Therefore in the stationary case Theorem 3.1 is the invariance principle of Newman and Wright [8] except the superflous third moment condition (2.3) .
(B) (3.2) and (3.4) imply condition (2.1) in Theorem 1 of Birkel [2] (see Lemmas 2 and 2 of [2]) and then (3.2), (3.3) and (3.4) imply condition (i) of Theorem 2 of [2].

Therefore the proof of Theoem 3.1 can be completed by Theorem 2 of Birkel [2].

REFERENCES

(1) Billingsley, P.(1968). *Convergence of Probability Measure.* Wiley, New york.

(2) Birkel(1988). The invariance principle for associated processes. *Stochastic Processes and Their Applcations,* 27, 57-71.

(3) Burton, R.M. and Kim, T.S.(1988). An invariance principle for associated random fields. *Pacific Journal of Mathmatics,* 132, 11-19.

(4) Cox, J.T. and Grimmet, G.(1984). Central limit theorems for associated random variables and percolation model. *The Annals of Probability,* 12, 514-528.

(5) Esary, J., Proschan, F. and Walkup, D.(1967). Association of random variables with applications. *Annals of Mathematical Statistics,* 38, 1466-1474.

(6) Newman, C.M.(1980). Normal fluctuations and the FKG inequalities. *Communications in Mathematical Physics,* 74, 119-128.

(7) Newman, C. M.(1984). Asymptotic independence and limit theorem for positively and negatively dependent random variables in:Y.L.Tong ed. Inequalities in Statistics and Probability. *The Institute of Mathematical Statistics*, Hayward, CA., 5, 127-140.

(8) Newman, C.M. and Wright, A.L.(1981). An invariance principle for certain dependent sequences. *The Annals of Probability*, 9, 671-675.

(9) Newman,C.M. and Wright,A.L.(1982). Associated random variables and martingale inequalities. *Zeitschrift für Wahrscheinlichkeits Theorie and Verwande Gebiete*, 59, 361-371.