A Note on the Value Approximation of Fuzzy Systems Variables

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ABSTRACT

Under the maxmin compositional rule of inference which is used in applications while executing fuzzy algorithms, Pappis showed that the property of approximation is preserved.

In this paper, we generalize a measure of proximity of fuzzy subsets on any set, without the restriction of finiteness. And it is shown that the same property of approximation is preserved under the supmin compositional rule of inference.

I. Introduction.

To model fuzzy logic controllers, it is necessary to approximate a set of linguistically expressed values of system variables and control statements to the corresponding set of fuzzy sets and relations. If we note that small deviations from what might be considered as 'precise membership values' should normally be of no practical significance[6], it is considerable to approximate the membership values.

In 1991, Pappis[2] introduced value approximation on a finite set U as follows.

Let A, A' be fuzzy subsets of $U=\{u_i \mid i=1, 2, \cdots, n\}$ and α_i, α_i' be the correponding grades of memberships, respectively. A and A' are said to be approximately equal (denoted by $A \approx A'$) if given a small nonnegative number $\varepsilon > 0$, $\max_i |\alpha_i - \alpha_i'| < \varepsilon$.

If B is another fuzzy subset of U, then $A \wedge B \approx A' \wedge B$ and $A \vee B \approx A' \vee B$. Also if and R' be fuzzy relations from to U to V, then $A \circ R \approx A' \circ R$ and $A \circ R \approx A \circ R'$.

In this note, we give value approximation on any set U. And we show that Pappis' properties also hold in this set U.

II. Proximity of fuzzy subsets

Let X be any set. Fuzzy subsets A and A' are said to be approximately equal, denoted by $A \approx A'$, if given a small number $\varepsilon > 0$, $\sup_{x \in X} |\mu(x) - \mu'(x)| \le \varepsilon$. Where μ and μ' are the membership functions of A and A', respectively. And the number ε is called a proximity measure of A and A'. Throughout this note, μ_A means the membership function of a fuzzy set A.

We need the following Lemmas.

Lemma 1. Let a_i, b_i be real numbers, $i=1, 2, \cdots, n$. Then

$$|\max(a_i) - \max(b_i)| \le \max|a_i - b_i|,$$

$$|\min(a_i) - \min(b_i)| \le \max|a_i - b_i|.$$

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Proof. Put $\max(a_i)=a$ and $\max(b_i)=b$. We can assume that $a>b$, and $a=a_i$, $b=b_j$. Then $|\max(a_i)-\max(b_i)|=a_i-b_j\le a_i-b_i\le \max|a_i-b_i|$.

The second inequality also can be proved similarly.

Lemma 2. Let f be bounded, real valued function on a set X. Then

(1) $|\sup_{x\in X} f(x)-\sup_{x\in X} g(x)|\le \sup_{x\in X}|f(x)-g(x)|$.

Proof. Set $\sup_{x\in X} f(x)=a$ and $\sup_{x\in X} g(x)=b$. Assume that $a>b$. Let $\varepsilon>0$ be given. Then there exists $x_0\in X$ such that $a-\varepsilon<f(x_0)$. Since $g(x_0)\le b$, $a-b<f(x_0)-g(x_0)+\varepsilon$. Thus $a-b\le \sup_{x\in X}|f(x)-g(x)|$.

Proposition 3. Let A_i and A_i' be fuzzy subsets of X, $i=1,\cdots,n$. If $A_i\approx A_i'$ for every i, then

(2) $\bigwedge_{i=1}^{n} A_i\approx \bigwedge_{i=1}^{n} A_i'$ and $\bigvee_{i=1}^{n} A_i\approx \bigvee_{i=1}^{n} A_i'$.

Proof. If we assume that A_i and A_i' are approximately equal, then $|\mu_{A_i}(x)-\mu_{A_i}'(x)|\le\varepsilon$, for all $x\in X$, which implies by Lemma 1 that $|\min_i(\mu_{A_i}(x))-\min_i(\mu_{A_i}'(x))|\le \max_i|\mu_{A_i}(x)-\mu_{A_i}'(x)|\le\varepsilon$, for all x, which follows that $\sup_{x\in X}|\min_i(\mu_{A_i}(x))-\min_i(\mu_{A_i}'(x))|\le\varepsilon$. Thus $\bigwedge_{i=1}^{n} A_i$ and $\bigwedge_{i=1}^{n} A_i'$ are approximately equal. For the unoin, we can prove similarly.

Directly from the proposition, we obtain the following corollary, which contains Pappis' properties 1 and 2 as the particular cases.

Corollary 4. If $A\approx A'$, $B\approx B'$ and $C\approx C'$, then $A\vee(B\wedge C)\approx A'\vee(B'\wedge C')$, and $A\wedge(B\vee C)\approx A'\wedge(B'\vee C')$.

Ⅲ. Fuzzy relations.

Let A and B be fuzzy subsets of X and let R be a fuzzy relation on $X\times Y$. The $\circ$-composition of A and B, denoted by $A\circ B$, is defined by the scalar

$A\circ B=\sup_{x\in X}\min(\mu_A(x),\ \mu_B(x))$.

And the $\circ$-composition of A and B, denoted by $A\circ R$, is defined by a fuzzy subset of Y with the corresponding membership function

$\mu_{A\circ R}(y)=\sup_{x\in X}\min(\mu_A(x),\ \mu_R(x,y))$.

Then the property of approximation preserves the above compositional rule of inference as follows.

Theorem 5. Let A, A' be fuzzy subsets of X, and let R, R' be the fuzzy relations on $X\times Y$. If $A\approx A'$ and $R\approx R'$, then $A\circ R\approx A'\circ R'$.

Proof. Let μ_R, μ_R' be the membership functions of R, R', respectively. Suppose that $\sup_{x\in X}|\mu_A(x)-\mu_A'(x)|\le\varepsilon$ and $\sup_{(x,y)\in X\times Y}|R(x,y)-R'(x,y)|\le\varepsilon$. By Lemma 1 and 2 for each fixed $y\in Y$,

$$
\begin{aligned}
&|\sup_{x\in X}\min(\mu_A(x),\ \mu_R(x,y))-\sup_{x\in X}\min(\mu_A'(x),\ \mu_R'(x,y))|\\
&\le \sup_{x\in X}|\min(\mu_A(x),\ \mu_R(x,y))-\min(\mu_A'(x),\ \mu_R'(x,y))|\\
&\le \sup_{x\in X}|\max\{|\mu_A(x)-\mu_A'(x)|,\ |\mu_R'(x,y)-\mu_R'(x,y)|\}\\
&<\varepsilon.
\end{aligned}
$$

It follows that

$$\sup_{y\in Y}|\mu_{A\circ R}(y)-\mu_{A'\circ R'}(y)|$$
$$=\sup_{y\in Y}|\sup_{x\in X}\min(\mu_A(x),\ \mu_R'(x,y))-\sup_{x\in X}\min(\mu_A'(x),\ \mu_R'(x,y)|$$
$$\leq \varepsilon,$$

which implies that $A\circ R \approx A'\circ R'$.

We have Pappis' main result[2] as a corollary of Theorem 5.

Corollary 6.

a) $A \approx A'$ implies $A\circ R \approx A'\circ R$.

b) $R \approx R'$ implies $A\circ R \approx A\circ R'$.

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