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항만 고유 진동 해석을 위한 Helmholtz 방정식의 유한요소 해법

Finite Element Solution of Helmholtz Equation for Free Harbor Oscillation

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Abstract

For the numerical analysis of free oscillation characteristics in a harbor with general boundary and bottom topography, finite element method is applied. The governing Helmholtz equation is transformed into a generalized matrix eigenvalue problem using the standard finite element procedure. A computer code is developed for the numerical evaluation of natural frequencies and free oscillation modes.

In the eigensolution process, a shifting strategy is devised for the treatment of numerical singularity. Scaling of coefficient matrix is also found to be effective for the alleviation of numerical ill-conditioning.

For the test problems, firstly, analytical and numerical solutions are compared and validity of the code is obtained. Hence the method is successfully applicable for the real-world problems with general geometric boundaries and bottom topography.

요 지

일반적인 기하학적 경계와 해저 지형을 가진 항만에서 해수 고유 진동 특성의 수치해석을 위하여 유한요소법이 응용되었다. 지배 방정식인 Helmholtz방정식을 일반화된 매트릭스 고유치문제로 변환하는데 표준유한요소과정을 사용하였다.

고유주기와 고유진동모우드의 수치해를 얻기위한 컴퓨터 프로그램이 개발되었고, 고유치의 수치해석과정에서 수치적 특이성을 취급하기 위해 고유치 이동기법이 고안되었으며, 수치적 악조건을 극복하기 위해서는 행렬원소의 축척화가 효과적임을 알았다.

수치예로서 먼저 해석해를 알 수 있는 경우를 해석하여 수치해와 해석해를 비교해 봄으로써 작성된 컴퓨터 프로그램의 유용성을 확인하였고, 일반적인 경계 조건과 임의 수심의 실제 항만에 유한요소 해법을 적용하여 성공적으로 고유진동의 해를 구하였다.

1. INTRODUCTION

Free oscillation characteristics in a harbor is one of the important factors for the develop-

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ment/design and the maintenance/operation of a harbor. It is closely related to the resonance and the forced oscillation under invading waves. The maximum wave amplitude or a resonance phenomenon within a harbor is greatly affected by natural periods and the solution of harbor oscillation may be represented as a linear combination of free oscillation modes. The long waves such as tides, storm surges and tsunamis can cause the harbor resonance if their periods are in coincidence with one of the free oscillation periods.[1]

For the analysis of free harbor oscillation, field observations or hydraulic and numerical model experiments have been used.[2] But the field observation and hydraulic model experiments cannot be pursued or are very expensive in case of newly-designed harbors. Therefore, analytical or numerical methods must be developed and some of them are shown to be applicable.[3] Moreover, recent advances in computer technology and development of numerical algorithms have made possible to apply the numerical methods and to treat various computational difficulties.

Some numerical methods ever effectively used may give unreasonable solutions due to the simplification of governing equations and the approximation of boundary conditions. Nevertheless, solutions of appropriate numerical methods are still better than those of simplified analytical ones. Such a case happens particularly in the problem with complex boundaries and bottom topography.

Among the various numerical methods, finite difference method and finite element method have been widely used for such a class of problems. But the finite difference approximation of governing equation has some difficulties when the geometric complexity of boundaries and bottom topography is involved.[4] In this regard, the finite element approach is shown to be robust. Ralston and Wilf[5] applied the finite element method in the free oscillation analysis of geometrically simple harbor, i.e., square basin with constant depth. However, other fields of finite element application related to the general characteristic analysis may be found. For example, Zienkiewicz[6] has performed eigenanalysis for arbitrary geometry in elasticity field. Mey[7] also presented an integral equation method for the eigenvalue problem in two-dimensional rectangular and circular electromagnetic field. But still in recent years, the application of finite element method on the free oscillation analysis in a harbor with general boundary and bottom topography conditions is in its infancy.

In this paper, an applicability of the finite element technique for the free oscillation analysis in a harbor with complex geometry and bottom topography is resumed and a general-purpose computer program is developed. To verify the validity of the method and the program, numerical examples are presented ; rectangular boundary with constant depth, rectangular boundary with variable depth, and real-world problems of arbitrary boundary and bottom topography.

2. GOVERNING EQUATION

According to the shallow water wave theory, the governing equation for the free harbor oscillation can be expressed as the Helmholtz equation in a harbor domain D;

$$c^2\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + \omega^2 F = 0 \tag{1}$$

where the boundary condition on the boundary ∂D is given as;

$$\frac{\partial F}{\partial n} = 0 \tag{9}$$

In Eqs.(1) and (2), c is phase velocity, and n unit outward normal vector at boundaries. The solutions of Eq.(1) with Eq.(2) are composed of eigenvalues ω^2 and corresponding eigenfunctions F(x,y) having physical meaning of natural frequencies squared, and the modes of the surface displacement in a harbor, respectively.

Since Eq.(1) is valid only under the assumption of constant water depth(h) through the domain, the following relations are held;

$$c = \sqrt{gh} \tag{3}$$

$$\omega = \frac{2\pi}{T} \tag{4}$$

where g is the gravitational acceleration and T the natural period.

In real harbors, h is no more constant. Thus Eq.(1) cannot be directly used for the free oscillation analysis in a harbor with general boundary and bottom topography. In this regard, an efficient and practically applicable numerical scheme to calculate free oscillation periods and modes in a real harbor should be provided.

3. FINITE ELEMENT FORMULATION

Galerkin's method, which is one of the weighted residual methods, is used to derive the finite element equation from Helmholtz equation (1). Here the eigenfunction F(x,y) is approximated through the interpolation of nodal values R_i as;

$$F(x,\ y)=\sum_{i=1}^{NN} N_i(x,\ y)R_i \tag{5}$$

where NN is the number of nodal points in an element, and Ni is properly chosen interpolation functions. When the linear triangular finite element is employed, then NN=3, which will be used in numerical examples later.

Substituting Eq.(5) into Eq.(1) and using Galerkin's procedure, one can derive the following element equation;

$$\sum_{i=1}^{NN}\int_{D(e)} N_j\{c^{(e)2}(\frac{\partial^2 N_i}{\partial x^2}+\frac{\partial^2 N_i}{\partial y^2})+\omega^2 N_i\}dD\ \ R_i^{(e)}=0 \tag{6}$$

where the superscript 'e' denotes the element-level quantities. In the derivation of element equation, depth of an element is kept constant for Eq.(6) or Eq.(1) to be valid, which may be the average of depths of nodal points. This is an only approximation of finite element procedure to solve the Helmholtz equation for the free harbor oscillation.

Integrating Eq.(6) by parts, we can obtain;

$$K_{ij}^{(e)}\ R_i^{(e)}-\omega^2\ M_{ij}^{(e)}\ R_i^{(e)}=Q_j^{(e)} \tag{7}$$

where

$$K_{ij}^{(e)}=c^{(e)2}\int_{D(e)}(\frac{\partial N_i}{\partial x}\ \frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\ \frac{\partial N_j}{\partial y})dD \tag{9}$$

$$M_{ij}^{(e)}=\int_{D(e)} N_i\ N_j\ dD \tag{9}$$

$$Q_j^{(e)}=c^{(e)2}\int_{B(e)} N_j(\frac{\partial}{\partial n}\sum_{i=1}^{NN} N_i R_i^{(e)})dB \tag{10}$$

In Eq.(10), $B^{(e)}$ is an element boundary, i.e., nodal points of an element, and the constant $c^{(e)}$ is defined as;

$$c^{(e)}=\sqrt{gh^{(e)}} \tag{11}$$

where $h^{(e)}$ is the average depth of element, which may be approximated as;

$$h^{(e)}=\frac{1}{NN}\sum_{i=1}^{NN}\ h_i^{(e)} \tag{12}$$

where $h_i^{(e)}$'s are the depths at the nodes of an element 'e'. As the element size decreases, the model bottom topography idealized with Eq.(12) approaches real situation.

Superposition of element equations, Eq.(7), with boundary condition, Eq.(2), leads to a generalized algebraic eigenvalue equations for the whole domain and boundary.

$$K\ R-\omega^2\ M\ R=0 \tag{13}$$

where $K=\sum_{e=1}^{NE}\ K^{(e)},\ M\sum_{e=1}^{NE}\ M^{(e)}$,

and NE is total number of elements. The matrix eigenproblem of Eq.(13) is solvable as matrices K and M must be positive definite or semidefinite.

4. OBSERVATIONS ON FINITE ELEMENET EQUATIONS

Critical observation on the finite element equation, Eq.(13), can be noted before the numerical solution procedure is established. They are related with the magnitudes of matrix elements and the inherent nature of matrices K and M of Eq.(13).

(1) Note that the orders of element magnitude in $K^{(e)}$ and $M^{(e)}$ of Eqs.(8) and (9) are respectively $O(h^{(e)}/A^{(e)})$ and $O(A^{(e)})$, where $A^{(e)}$ is the area of a finite element. Hence the difference of magnitude in $K^{(e)}$ and $M^{(e)}$ may become prohibitively significant in case of a large domain with relati-

vely shallow depth. As a matter of fact, it would cause a numerical difficulty when the matrix eigenproblem is numerically solved. Thus an appropriate scaling of coefficient matrices in Eq.(13) should be employed in the numerical computation.

(2) In the governing eigenproblem of Helmholtz equation, Neumann type boundary condition of Eq.(2) has been posed, which implies a zero eigenvalue of 2 or rigid body modes of eigenfunction F. Such a phenomenon will be directly represented in the matrix eigenproblem of Eq.(13) and resulting matrices K and M become at most positive semidefinite or singular. Thus, in the numerical implementation of solution procedure, such a singularity problem should be specially handled.

5. SOLUTION SCHEMES OF FINITE ELEMENT EQUATIONS

A variety of numerical methods for the solution of generalized algebraic eigenvalue problems is available.[8,9] Among them the subspace iteration method is suitable for the partial eigensolution of large scale problems such as Eq.(13).

In the method, matrices K and M of Eq.(13) are supposed to be positive definite and at least positive semi-definite, respectively. However, the matrix K in Eq.(13) does not satisfy the positive-definite condition since it contains the rigid body mode due to the Neumann type boundary conditions of Eq.(2). Thus it is singular in nature. The singularity of matrix K usually causes the numerical difficulties in the numerical eigensolution procedure and an appropriate technique must be employed. Furthermore, the numerical ill-conditioning problem will happen in the numerical computation when the elements of matrix K become much smaller than those of matrix M. Such a situation arises when the ratio of depth to width of harbor is significantly small, which is usually the case.

To alleviate numerical ill-conditioning, scaling of matrices is introduced in Eq.(13) as;

$$M^* = 10^{-s} M \tag{14}$$

$$\mu = 10^{s} \omega^2 \tag{15}$$

$$K\ R = \mu\ M^*\ R \tag{16}$$

where s is a scaling factor. This way the orders of magnitude in the elements of matrices K and M^* are made approximately the same. Hence the numerical ill-conditioning can be avoided.

For the treatment of singularity, shifting strategy is effectively applied in Eq.(16) as;

$$K^* = K + \sigma\ M^* \tag{17}$$

$$\lambda = \mu + \sigma \tag{18}$$

$$K^*\ R = \lambda\ M^*\ R \tag{19}$$

where σ is a shifting parameter and λ shifted eigenvalues. Shifting parameter σ must be so chosen that K^* is no longer singular matrix, and the original eigenvalues can be recovered as;

$$\omega^2 = 10^{-s}(\lambda - \sigma) \tag{20}$$

Note that the eigenvectors are not affected with shifting and scaling.

6. NUMERICAL EXAMPLES

Three types of numerical examples are presented herein for the purpose of testing the developed numerical method and examining the applicability on real-world problems. For the former, a problem of square boundary and uniform bottom topography or constant water depth is taken, for which analytical solutions are also available. They are compared with finite element solutions. The second example is posed as variable depth and square boundary. The third one is a real harbor with variable depth and general boundary. They are used to examine the applicability and some qualitative discussions on their solutions will be possible.

For all the numerical examples presented, shifting factor, $\sigma = 0.5$, is used in the computation. Scaling factor is set to zero($s = 0.0$) in Examples 1 and 2, while $s = 10.0$ is used for Example 3.

6.1 Example 1: Square basin with constant depth

It is a square basin of 160x160m with constant depth of 4m. Analytical solutions for rectangular basins with constant depth have been found as[1,10,11];

$$\omega^2 = c^2\pi^2\left(\frac{k^2}{a^2} + \frac{m^2}{b^2}\right);\ k,\ m = 0,\ 1,\ 2\cdots \tag{21}$$

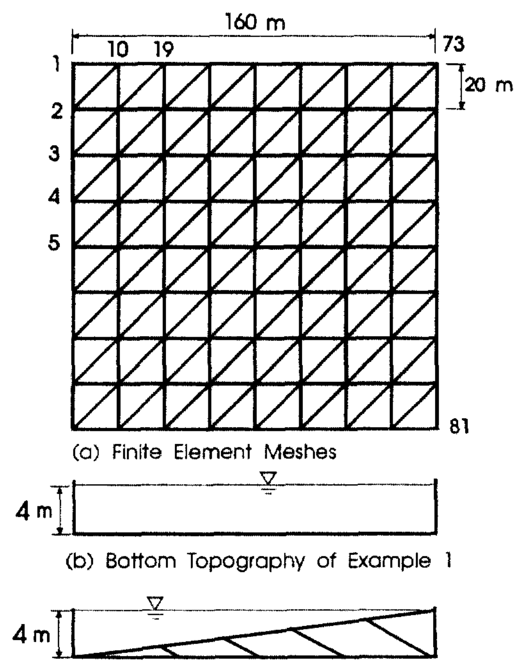
$$F(x,\ y) = \cos\left(\frac{k\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right) \tag{22}$$

where a and b are the sides of rectangular boundary in x and y direction respectively. Since a=b in the example, according to Eq.(21) and Eq.(22), numerical solutions of pairs of the same eigenvalues with corresponding symmetric modes are expected. Moreovereigenfunctions F(x,y) or numerical eigenvectors R do not have to depend on the depth h once it is constant.

Finite element system for numerical computation has 81 degrees of freedom and 128 linear triangular elements as shown in Fig. 1. Therefore, the size of matrices and consequently the number of eigensolutions must be 81, the total number of nodal points. Analytical and numerical finite element solutions for the lowest few modes are presented in Table 1 and Fig. 2 which shows the contours of eigenvectors. Numerical solutions of natural periods are shown to have reasonable accuracy compared with the analytical ones(less than 2.5 % error). Corresponding eigenvectors associated with the same eigenvalues show the similar shapes although the amplitudes of eigenvectors are symmetrically reversed, as can be seen in Fig. 2. Thus the usability of the developed finite element program is found.

Table 1. Free Oscillation Periods of Example 1 and 2

Mode	Example 1, sec.		Example 2, sec.
	Analytical	Numerical (% Error)	Numerical
0	∞	1.2×10^{12}	1.2×10^{8}
1	50.60	50.28(0.63)	89.53
2	50.60	50.28(0.63)	82.56
3	35.78	35.12(1.84)	59.33
4	25.30	24.68(2.45)	54.50
5	25.30	24.67(2.45)	45.27



(a) Finite Element Meshes

(b) Bottom Topography of Example 1

(c) Bottom Topography of Example 2

Fig. 1. Finite Element Meshes and Bottom Topography of Example 1 and Example 2.

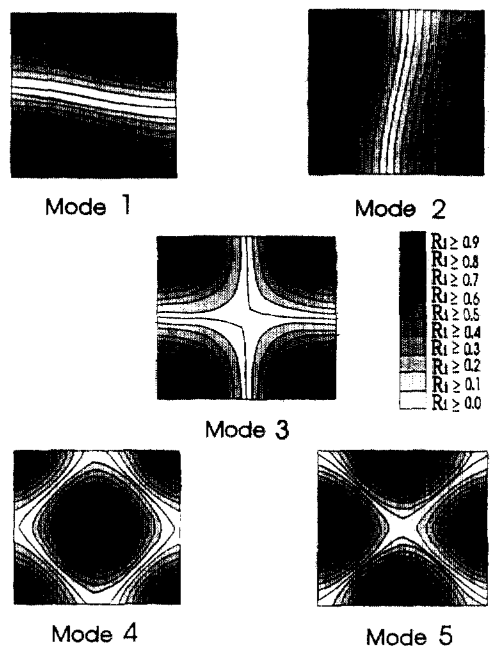


Fig. 2. Normalized Eigenvectors for Example 1.

6.2 Example 2: Square basin with variable depth

It has the same boundary as Example 1 but the uniform bottom slope of 0.025 as shown in Fig. 1. Eigenvalues are presented in Table 1. Natural periods of Example 2 are consistently larger than those of Example 1 at the same mode, which accounts for the effect of bottom topography. The contours of eigenvectors are presented in Fig. 3. As can be seen in the figure, the amplitudes of eigenvectors at shallow region are relatively larger than those in the deep region. This agrees with Hidaka's result stating that the wave amplitudes of the shallow waters are larger than those of deep ones.[10,12] Thus, the validity of the program is verified qualitatively as well as quantitatively.

6.3 Example 3: Real harbor with general boundary and depth

For the verification of applicability of finite element procedure, an example of real harbor is considered. Boundary and bottom topography of study area is shown in Fig. 4. Harbor boundary is quite general and water depths are in the range of 1.0~25.0m.

Construction of artificial island is planned in the region as an initial stage of The New Marine Town Project of Pusan, Korea. Since the free oscillation characteristics of the region before and after the construction of artificial island are of interest, two cases are considered ; with and without artificial island. Finite element idealizations for both cases(Case I without island ; Case II with island) are respectively shown in Fig. 5 and 6 where Case I has 237 nodes and 441 elements and Case II 211 nodes and 311 elements.

Numerical results of oscillation periods are presented in Table 2 and the normalized eigenvec-

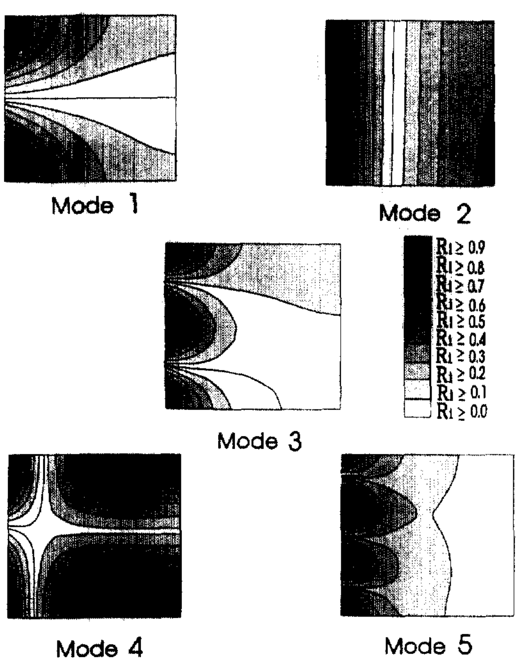


Fig. 3. Normalized Eigenvectors for Example 2.

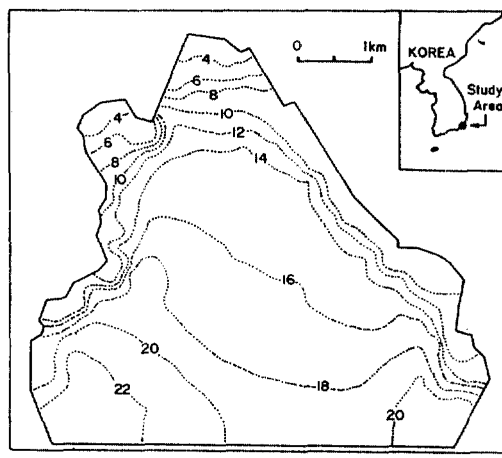


Fig. 4. Domain and Boundary of Example 3.

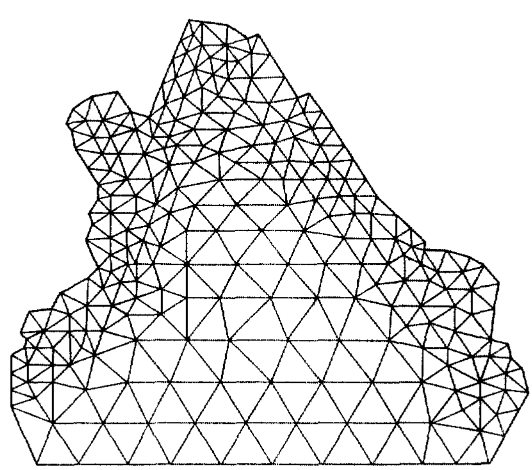


Fig. 5. Finite Element Mesh of Example 3(Case I).

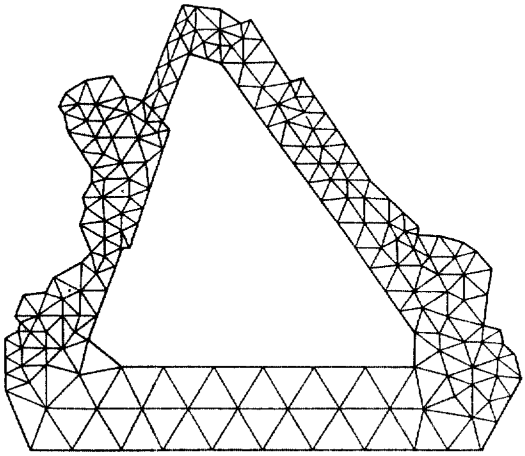


Fig. 6. Finite Element Mesh of Example 3(Case II).

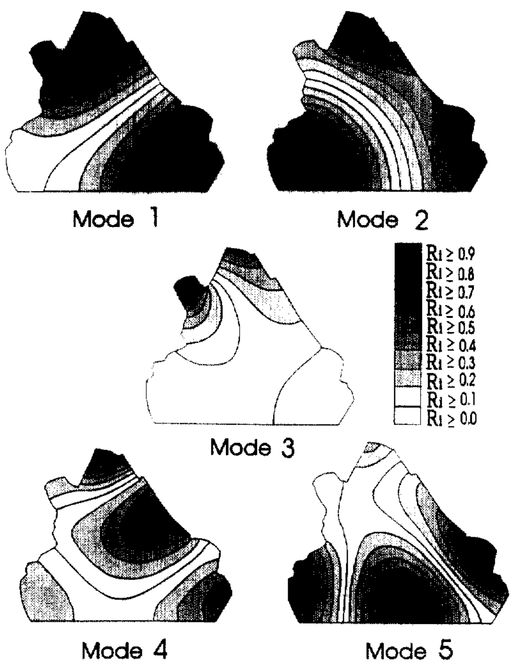


Fig. 7. Normalized Eigenvectors for Example 3 (Case I).

tors, $|R_i| \leq 1.0$, are in Figs. 7 and 8 respectively. The periods of no island case become much shorter than those of the case with island. Maximum amplitudes of eigenvectors consistently appear at shallow regions for both cases. Such a trend is very reasonable representation of Hidaka's result.

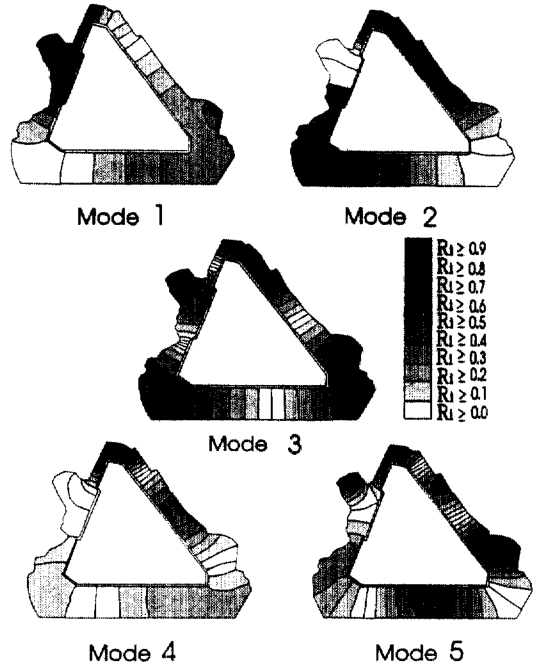


Fig. 8. Normalized Eigenvectors for Example 3 (Case II).

Table 2. Finite Element Solution of Example 3

Mode	Free Oscillation Periods, min.	
	Case I : Without Island	Case II : With Island
1	22.86	36.75
2	19.77	35.21
3	13.72	21.47
4	12.15	16.02
5	10.29	12.53

Thus an applicability of the method and program is assured.

7. SUMMARY AND CONCLUSION

Finite element procedure for the analysis of free harbor oscillation with a general boundary and bottom topography is described. The governing Helmholtz equation is transformed into a generalized algebraic eigenvalue problem. Shifting strategy and scaling factor are introduced in the solution process to treat singularity and numerical ill-conditioning of the problem.

For the verification of validity and applicability of the method and finite element program, three numerical examples are solved. As for the validity and accuracy, available analytical solutions are compared with numerical solutions. They are in excellent agreement. Qualitative trends of finite element solution are also found to be reasonable, which can be drawn based on the Hidaka's result.

From the study and numerical results, it is concluded that developed numerical schemes and finite element solutions are very reasonable. Hence the usability of finite element method for the practical analysis of free harbor oscillation with general boundary and bottom topography is proved.

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