大韓土木學會論文集
第13卷 第4號 · 1993年 9月
pp. 151~161

水 工 學

對數-Gumbel 確率分布函數의 媒介變數 推定과 信賴限界 誘導

Parameter Estimation and Confidence Limits for the Log-Gumbel Distribution

허 준 행*
Heo, Jun Haeng

Abstract

The log-Gumbel distribution in real space is defined by transforming the conventional log-Gumbel distribution in log space. For this model, the parameter estimation techniques are applied based on the methods of moments, maximum likelihood and probability weighted moments. The asymptotic variances of estimator of the quantiles for each estimation method are derived to find the confidence limits for a given return period. Finally, the log-Gumbel model is applied to actual flood data to estimate the parameters, quantiles and confidence limits.

요 지

본 연구에서는 기존의 對數형태인 對數-Gumbel 확률분포함수를 변환하여 새로운 형태의 對數-Gumbel 확률분포함수를 정립하였다. 이 분포함수를 이용하여 모멘트법, 최우도법, 확률가중모멘트법(Probability weighted moments)에 기초한 매개변수 추정과정을 유도하였으며, 또한 재현기간별 신뢰한계를 구하기 위하여 각각의 매개변수 추정법에 대한 漸近分散式을 유도하였다. 아울러 유도된 식들을 실제 자료에 적용하였다.

1. Introduction

The log-Gumbel distribution is one of the commonly used distributions for frequency analysis in hydrology. In the literature, the log-Gumbel distribution is also known as the Frechet distribution[1]. The importance of using the Gumbel and log-Gumbel distributions was indicated by Shen et al.[2] and Ochoa et al.[3] They studied the effect of the tail behavior assumptions of these distributions for fitting the annual floods of more than 200 stations in Texas, New Mexico and Colorado. They concluded that the log-Gumbel distribution provided a better fit for more than two-thirds of the total stations and also showed a greater estimates of extreme flood magnitude than the Gumbel distribution.

The log-Gumbel distribution is a special case of the Generalized Extreme Value (GEV) distribution, especially the GEV-2 distribution. Prescott and Walden[4] derived the expected values of the second order derivatives of log-likelihood functions of the GEV distribution with respect to the parameters. Later, Prescott and Walden[5] showed the iterative estimation procedure for the maximum likelihood estimates of the GEV distribution and derived the observed information matrix of

* 정회원 · 콜로라도 주립대학교 토목공학과 연구원, 공학박사

the censored samples as a reasonable approximation for the maximum likelihood estimates.

Hosking et al.[6,7] estimated parameters of the GEV distribution based on the method of probability weighted moments (PWM) and gave the asymptotic variances of the parameters and table values of the asymptotic variance of the PWM quantile estimator. Recently, Liu and Stedinger[8] compared the variances of the PWM quantile estimator for the two and three parameter GEV distributions. Although there is a relationship between the log-Gumbel and GEV models, there is few studies for the log-Gumbel model, especially for the confidence limits. The purposes of this study are to introduce the parameter estimation techniques and to derive the asymptotic variances of estimator of the quantile to obtain the confidence limits for the log-Gumbel distribution in real space. The second chapter of this paper defines the log-Gumbel model in real space by transforming the conventional log-Gumbel model in log space, and describes the relationships between this model and the GEV model, and statistical properties of the log-Gumbel distribution. The methods of moments (MOM), maximum likelihood (ML) and probability weighted moments (PWM) are proposed to estimate the parameters of the log-Gumbel distribution in the third chapter. To obtain the confidence limits of the MOM, ML and PWM quantile estimators, the asymptotic variances of the corresponding quantile estimator are derived for a given return period and parameters in the fourth chapter. Finally, the log-Gumbel model is applied to the annual flood data of the St. Mary's River in Canada.

2. Model Description

Consider that random variables Y and X are related as $Y=\log(X-x_o)$ in which log represents the natural logarithm. It may be shown that Y is Gumbel distributed with location parameter y_o and scale parameter α, if X is log-Gumbel distributed with parameters x_o, y_o and α. Thus, the cumulative distribution function (CDF) of the log-Gumbel distribution is given by

$$F(x)=\exp\left\{-\exp\left[-\frac{\log(x-x_o)-y_o}{\alpha}\right]\right\} \qquad (1)$$

for $x>x_o$ and $\alpha>0$.

Likewise, the log-Gumbel distribution is related to the GEV-2 distribution. For instance, by assuming that $x_o=x_o'+\alpha'/\beta'$, $\alpha=-\beta'$ and $y_o=\log(-\alpha'/\beta')$ it may be shown that the CDF given by Eq. (1) can be written in the form of CDF of the GEV-2 distribution (1) as

$$F(x)=\exp\left\{-\left[1-\frac{\beta'(x-x_o')}{\alpha'}\right]^{1/\beta'}\right\} \qquad (2)$$

in which x_o', α' and β' are the location, scale and shape parameters of such GEV-2 distribution and the shape parameter β' is negative. Furthermore, assuming that $\alpha=1/\beta$ and $y_o=\log(\theta-x_o)$, it may be shown that Eq. (1) takes the form

$$F(x)=\exp\left[-\left(\frac{\theta-x_o}{x-x_o}\right)^{\beta}\right] \qquad (3)$$

in which $\theta>x_o$, $\beta>0$ and $x_o<x<\infty$. Equation (3) is another form of the log-Gumbel distribution. In addition, it may be also shown that by assuming $\beta=-1/\beta'$, $\theta=x_o'$ and $x_o=x_o'+\alpha'/\beta'$, the CDF of Eq. (3) takes the form of the CDF of the GEV-2 distribution of Eq. (2). In the remainder of this paper, we will use the log-Gumbel model given by Eq. (3) to derive other properties of this distribution.

The probability density function (PDF) of the log-Gumbel distribution is given by

$$f(x)=\frac{\beta}{(x-x_o)}\left(\frac{\theta-x_o}{x-x_o}\right)^{\beta}\exp\left[-\left(\frac{\theta-x_o}{x-x_o}\right)^{\beta}\right] \qquad (4)$$

Figure 1 shows some typical shapes of the PDF of the log-Gumbel distribution.

The rth moment of X around x_o can be shown to be

$$E[(X-x_o)^r]=(\theta-x_o)^r\Gamma(1-r/\beta) \qquad (5)$$

where $\Gamma(W)$ is the gamma function with argument w. The mean and variance can be obtained as

$$\mu=x_o+(\theta-x_o)\Gamma(1-1/\beta) \qquad (6)$$

and

$$\sigma^2=(\theta-x_o)^2[\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)] \qquad (7)$$

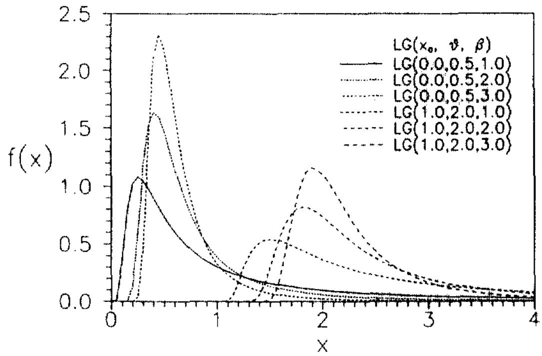

Fig. 1. The Probability Density Functions of the Log-Gumbel Distribution

Note that the mean will exist for $\beta>1$ and the variance will exist for $\beta>2$. Likewise, the skewness coefficient is given by

$$\gamma=\frac{\Gamma(1-3/\beta)-3\Gamma(1-2/\beta)\Gamma(1-1/\beta)+2\Gamma^3(1-1/\beta)}{[\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)]^{3/2}} \tag{8}$$

for $\beta>3$. Note that the skewness coefficient of the log-Gumbel distribution is greater than 1.1396. Additionally, the mode is given by

$$\text{mode}(x)=x_o+(\theta-x_o)\left[\frac{1+\beta}{\beta}\right]^{-1/\beta} \tag{9}$$

3. Estimation of Parameters

Three estimation procedures are presented for the log-Gumbel distribution; method of moments, method of maximum likelihood and method of probability weighted moments.

3.1 Method of Moments

By substituting μ, σ and γ in Eqs. (6), (7) and (8) for corresponding sample estimates $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\gamma}$, the method of moments estimators $\hat{\theta}$, $\hat{\beta}$ and $\hat{x}_o$ can be obtained. The skewness coefficient in Eq. (8) is only a function of the shape parameter β. . Thus, the moment estimator of the shape parameter, $\hat{\beta}$ can be obtained from the approximate polynoimal equations derived by

$$\hat{\beta}=222.5222-313.1802\hat{\gamma}+179.5053\hat{\gamma}^2-50.6058\hat{\gamma}^3+6.978542\hat{\gamma}^4-0.376228\hat{\gamma}^5 \tag{10a}$$

which is valid for $1.48<\hat{\gamma}<5.4$ and

$$\hat{\beta}=1731.6756-2342.8143\hat{\gamma}+1802.1566\hat{\gamma}^2 \tag{10b}$$

valid for $1.1396<\hat{\gamma}\leq 1.48$ in which $\hat{\gamma}$ is the sample skewness coefficient. For a more precise solution of $\hat{\gamma}$, Eq. (10) can be used as the initial value for a numerical procedure such as Newton-Raphson method. For this purpose, Eq. (8) is rewritten as

$$\gamma=\frac{\Gamma(1-3/\hat{\beta})-3\Gamma(1-1/\hat{\beta})\Gamma(1-1/\hat{\beta})+2\Gamma^3(1-1/\hat{\beta})}{[\Gamma(1-2/\hat{\beta})-\Gamma^2(1-1/\hat{\beta})]^{3/2}} \tag{11}$$

and the first derivative of Eq. (11) with respect to $\hat{\beta}$ is given by

$$\begin{aligned}F'(\hat{\beta})=&\frac{1}{\hat{\beta}^2[\Gamma(1-2/\hat{\beta})-\Gamma^2(1-1/\hat{\beta})]^{5/2}}\times\\&\{[3\Gamma'(1-3/\hat{\beta})-6\Gamma'(1-2/\hat{\beta})\Gamma(1-1/\hat{\beta})-3\Gamma'(1-1/\hat{\beta})\Gamma(1-2/\hat{\beta})+6\Gamma'(1-1/\hat{\beta})\Gamma^2(1-1/\hat{\beta})]\\&[\Gamma(1-2/\hat{\beta})-\Gamma^2(1-1/\hat{\beta})]-[\Gamma(1-3/\hat{\beta})-3\Gamma(1-2/\hat{\beta})\Gamma(1-1/\hat{\beta})+2\Gamma^3(1-1/\hat{\beta})][3\Gamma'(1-2/\hat{\beta})-3\Gamma'(1-1/\hat{\beta})\Gamma(1-1/\hat{\beta})]\}\end{aligned} \tag{12}$$

where $\Gamma'(w)$ is the first derivative of the gamma function with argument w. Thus, the recursive equation to estimate $\hat{\beta}$ at the iteration $i+1$ is given by

$$\hat{\beta}_{i+1}=\hat{\beta}_i-F(\hat{\beta}_i)/F'(\hat{\beta}_i) \tag{13}$$

and is repeated until satisfying the following error criterion

$$\left|\frac{\hat{\beta}_{i+1}-\hat{\beta}_i}{\hat{\beta}_i}\right|<\varepsilon \tag{14}$$

in which ε is a specified relative error.

Once $\hat{\beta}$ is determined from Eq. (13), $\hat{x}_o$ is obtained by combining Eqs. (6) and (7) as

$$\hat{x}_o=\hat{\mu}-\frac{\hat{\sigma}\Gamma(1-1/\hat{\beta})}{[\Gamma(1-2/\hat{\beta})-\Gamma^2(1-1/\hat{\beta})]^{1/2}} \tag{15}$$

Finally, $\hat{\theta}$ is determined from Eq. (6)

$$\hat{\theta}=\hat{x}_o+\frac{\hat{\mu}-\hat{x}_o}{\Gamma(1-1/\hat{\beta})} \tag{16}$$

Note that the moment estimators will exist for

$\beta>3$ because the population moments exist depending on the values of the shape parameter.

For a two parameter log-Gumbel distribution ($x_o=0$), $\hat{\beta}$ can be obtained numerically from Eq. (15). Once $\hat{\beta}$ is obtained, $\hat{\theta}$ is determined from Eq. (16).

3.2 Method of Maximum Likelihood

The log-likelihood function of the three parameter log-Gumbel distribution is given by

$$LL(\underline{x};x_o,\ \beta,\ \theta)=N\log(\beta)+N\beta\log(\theta-x_o) -(\beta+1)\sum_{i=1}^{N}\log(x_i-x_o) -\sum_{i=1}^{N}\left[\frac{\theta-x_o}{x_i-x_o}\right]^{\beta} \tag{17}$$

where N is sample size. Taking partial derivatives of the log-likelihood function with respect to x_o, β and θ, respectively, and equating each to zero give

$$\partial LL/\partial x_o=-\frac{N\beta}{\theta-x_o}+(\beta+1)\sum_{i=1}^{N}(x_i-x_o)^{-1} +\beta(\theta-x_o)^{\beta-1}\sum_{i=1}^{N}(x_i-x_o)^{-\beta} -\beta(\theta-x_o)^{\beta}\sum_{i=1}^{N}(x_i-x_o)^{-(\beta+1)}=0 \tag{18}$$

$$\partial LL/\partial\beta=\frac{N}{\beta}+N\log(\theta-x_o)-\sum_{i=1}^{N}\log(x_i-x_o) -\sum_{i=1}^{N}\left[\frac{\theta-x_o}{x_i-x_o}\right]^{\beta}\log\left[\frac{\theta-x_o}{x_i-x_o}\right]=0 \tag{19}$$

$$\partial LL/\partial\theta=\frac{N\beta}{\theta-x_o}-\beta(\theta-x_o)^{\beta-1}\sum_{i=1}^{N}(x_i-x_o)^{-\beta}=0 \tag{20}$$

These equations should be solved simultaneously to find the estimators of the parameters x_o, β and θ. The increments of the parameters x_o, β and θ based on the Newton-Raphson method can be written as

$$\begin{bmatrix}\Delta x_o\\ \Delta\theta\\ \Delta\beta\end{bmatrix}=\begin{bmatrix}-\partial^2LL/\partial x_o^2 & -\partial^2LL/\partial x_o\partial\theta & -\partial^2LL/\partial x_o\partial\beta\\ -\partial^2LL/\partial\theta\partial x_o & -\partial^2LL/\partial\theta^2 & -\partial^2LL/\partial\theta\partial\beta\\ -\partial^2LL/\partial\beta\partial x_o & -\partial^2LL/\partial\beta\partial\theta & -\partial^2LL/\partial\beta^2\end{bmatrix}^{-1}\begin{bmatrix}\partial LL/\partial x_o\\ \partial LL/\partial\theta\\ \partial LL/\partial\beta\end{bmatrix} \tag{21}$$

where -1 represents the inverse of matrix, Δx_o, $\Delta\theta$ and $\Delta\beta$ are the increments of the parameters x_o, θ and β, respectively, and the second partial derivatives of the log-likelihood function of the log-Gumbel distribution are given in Appendix A. Therefore, the new estimates at the iteration (i+1) are computed by

$$\lambda_{i+1}=\lambda_i+\Delta\lambda_i \tag{22}$$

until satisfying the error criterion

$$|\ \Delta\lambda_i/\lambda_{i+1}\ |<\varepsilon \tag{23}$$

in which λ represents one of the parameters x_o, θ and β, and ε is a specified relative error.

3.3 Method of Probability Weighted Moments

The general form of the probability weighted moments (PWM) defined by Greenwood et al.[9] is applied to the three parameter log-Gumbel distribution as

$$B_r=E[X(F(x))^r] =\frac{1}{r+1}[x_o+(\theta-x_o)(r+1)^{1/\beta}\Gamma(1-1/\beta)] \tag{24}$$

which is valid for $\beta>1$ and r is a nonnegative integer. Using the first three PWMs (r=0, 1, 2), the PWM estimators of the log-Gumbel distribution are given by

$$\frac{1-3^{1/\hat{\beta}}}{1-2^{1/\hat{\beta}}}=\frac{3\hat{B}_2-\hat{B}_0}{2\hat{B}_1-\hat{B}_0} \tag{25}$$

$$\hat{\theta}=\hat{B}_0+\frac{\Gamma(1-1/\hat{\beta})-1}{(1-2^{1/\hat{\beta}})\Gamma(1-1/\hat{\beta})}(2\hat{B}_1-\hat{B}_0) \tag{26}$$

$$\hat{x}_o=\frac{2^{1/\hat{\beta}}\hat{B}_0-2\hat{B}_1}{(2^{1/\hat{\beta}}-1)} \tag{27}$$

in which the sample PWM $\hat{B}_r$ is given by Hosking et al.[6]

$$\hat{B}_0=\frac{1}{N}\sum_{j=1}^{N}x_j \qquad \text{for } r=0 \tag{28a}$$

$$\hat{B}_r=\frac{1}{N}\sum_{j=1}^{N}x_j\frac{(j-1)(j-2)\cdots(j-r)}{(N-1)(N-2)\cdots(N-r)} \quad \text{for } r>1 \tag{28b}$$

where x_j is the order statistic such that $x_1\leq x_2\leq\cdots\leq x_N$. The PWM estimator of the shape parameter, $\hat{\beta}$ can be obtained numerically by the Newton-Raphson method. The value of $\hat{\beta}$ from Eq. (10) can be used as the initial value to solve Eq. (25) for $\hat{\beta}$.

For a two parameter log-Gumbel distribution ($x_o=0$), the PWM estimators can be obtained from the first two PWMs ($r=0$, 1)

$$\hat{\beta}=\log(2)/\log(2\hat{B}_1/\hat{B}_0) \tag{29}$$

and

$$\hat{\theta}=\hat{B}_0/\Gamma(1-1/\hat{\beta}) \tag{30}$$

Note that for a two parameter case, the PWM estimates can be determined directly without any iterative procedure.

4. Confidence Limits of Estimator of Quantiles

The $\gamma=(1-\alpha)$ confidence limits X_l on the population quantiles may be determined by

$$X_l=\hat{X}_T\pm u_{1-\alpha/2}S_T \tag{31}$$

where $u_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution, $\hat{X}_T$ is the quantile estimator corresponding to return period T, and S_T is the standard deviation of $\hat{X}_T$. The quantile estimator $\hat{X}_T$ of the log-Gumbel distribution can be obtained from Eq. (3) as

$$\hat{X}_T=\hat{x}_o+(\hat{\theta}-\hat{x}_o)[-\log(1-1/T)]^{-1/\hat{\beta}} \tag{32}$$

where the cumulative distribution function, F(x) is replaced by $(1-1/T)$. Also, the estimator $\hat{X}_T$ may be generally written in terms of sample mean $\hat{\mu}$, sample standard deviation $\hat{\sigma}$ and the frequency factor K_T[10]

$$\hat{X}_T=\hat{\mu}+K_T\hat{\sigma} \tag{33}$$

4.1 Standard Error by Moments

The variance of $\hat{X}_T$ can be expressed as[11]

$$\begin{aligned}S_T^2=\mathrm{Var}(\hat{X}_T)=\frac{\mu_2}{N}\{&1+K_T\gamma+K_T^2(\gamma_2-1)/4\\&+(\partial K_T/\partial\gamma)[2\gamma_2-3\gamma^2-6+K_T(\gamma_3-6\gamma\gamma_2/4-10\gamma/4)]\\&+(\partial K_T/\partial\gamma)^2[\gamma_4-3\gamma\gamma_3-6\gamma_2+9\gamma^2\gamma_2/4+35\gamma^2/4+9]\}\end{aligned} \tag{34}$$

where μ_r is a rth central moment and

$$\gamma=\mu_3/\mu_2^{3/2} \tag{35a}$$

$$\gamma_2=\mu_4/\mu_2^{3} \tag{35b}$$

$$\gamma_3=\mu_5/\mu_2^{5/2} \tag{35c}$$

$$\gamma_4=\mu_6/\mu_2^{3} \tag{35d}$$

The derivative of K_T with respect to γ can be written as

$$(\partial K_T/\partial\gamma)=(\partial K_T/\partial\beta)(\partial\beta/\partial\gamma) \tag{36}$$

Substituting Eqs. (6) and (7) into Eq. (33) and solving for K_T yield

$$K_T=\frac{[-\log(1-1/T)]^{-1/\beta}-\Gamma(1-1/\beta)}{[\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)]^{1/2}} \tag{37}$$

then the derivative of K_T with respect to β is given by

$$\left(\frac{\partial K_T}{\partial\beta}\right)=[(-D_1\psi_1+\log S S^{-1/\beta})(D_2-D_1^2)-(S^{-1/\beta}-D_1)(D_2\psi_2-D_1^2\psi_1)]/\beta^2(D_2-D_1^2)^{3/2} \tag{38}$$

where $D_r=\Gamma(1-r/\beta)$, $\psi_r=\psi(1-r/\beta)=\Gamma'(1-r/\beta)/\Gamma(1-r/\beta)$ and $S=-\log(1-1/T)$. The derivative of γ with respect to β can be obtained from Eq. (8) as

$$\begin{aligned}(\partial\gamma/\partial\beta)=3[&(D_2-D_1^2)(D_3\psi_3-2D_2D_1\psi_2-D_2D_1\psi_1\\&+2D_1^3\psi_1)+(D_3-3D_2D_1+2D_1^3)(D_2\psi_2\\&-D_1^2\psi_1)]/[\beta^2(D_2-D_1^2)^{5/2}]\end{aligned} \tag{39}$$

Thus, Eq. (36) can be obtained from Eq. (38) and the reciprocal of Eq. (39).

To find the cumulants γ, γ_2, γ_3 and γ_4 in Eq. (35), the central moments of the log-Gumbel distribution are given by

$$\mu_2=(\theta-x_o)^2[D_2-D_1^2] \tag{40a}$$

$$\mu_3=(\theta-x_o)^3[D_3-3D_2D_1+2D_1^4] \quad (40b)$$

$$\mu_4=(\theta-x_o)^4[D_4-4D_3D_1+6D_2^{12}-3D_1^4] \quad (40c)$$

$$\mu_5=(\theta-x_o)^5[D_5-5D_4D_1+10D_3^{12}-10D_2D_1^3 +4D_1^5] \quad (40d)$$

$$\mu_6=(\theta-x_o)^6[D_6-6D_5D_1+15D_4^{12}-20D_3D_1^3 +15D_2D_1^4-5D_1^6] \quad (40e)$$

Finally, the asymptotic variance of $\hat{X}_T$ of Eq. (34) is obtained from Eqs. (35) through (40).

Note that the frequency factors K_T of Eq. (37) is a function of skewness coefficient and return period. Therefore, the frequency factor values can be obtained for given skewness coefficients and return periods as shown in Table 1. Hence, the quantile $\hat{X}_T$ can be easily obtained from Eq. (33) using Table 1 when the sample mean, standard deviation and skewness coefficient are known.

4.2 Standard Error by Maximum Likelihood

The asymptotic variance of the estimator of quantile, $\hat{X}_T$, for the three parameter log-Gumbel distribution is given by[8]

$$S_T^2=\left(\frac{\partial X_T}{\partial x_o}\right)^2 Var(\hat{x}_o)+\left(\frac{\partial X_T}{\partial \theta}\right)^2 Var(\hat{\theta}) +\left(\frac{\partial X_T}{\partial \beta}\right)^2 Var(\hat{\beta})+2\left(\frac{\partial X_T}{\partial x_o}\right)\left(\frac{\partial X_T}{\partial \theta}\right) Cov(\hat{x}_o,\ \hat{\theta})+2\left(\frac{\partial X_T}{\partial \theta}\right)\left(\frac{\partial X_T}{\partial \beta}\right)Cov(\hat{\theta},\ \hat{\beta}) +2\left(\frac{\partial X_T}{\partial x_o}\right)\left(\frac{\partial X_T}{\partial \beta}\right)Cov(\hat{x}_o,\ \hat{\beta})$$

in which the derivatives of X_T with respect to the parameters x_o, θ and β are given by

$$\left(\frac{\partial X_T}{\partial x_o}\right)=1-[-\log(1-1/T)]^{-1/\beta} \quad (42a)$$

$$\left(\frac{\partial X_T}{\partial \theta}\right)=[-\log(1-1/T)]^{-1/\beta} \quad (42b)$$

$$\left(\frac{\partial X_T}{\partial \beta}\right)=-\frac{\theta-x_o}{\beta^2}\log[-\log(1-1/T)]^{-1/\beta} [-\log(1-1/T)]^{-1/\beta} \quad (42c)$$

and

$$Var(\hat{x}_o)=(\theta-x_o)^2\{1+\Gamma''(2)-[\Gamma'(2)]^2\}/N\beta^2D \quad (43a)$$

Table 1. Frequency Factors for the Log-Gumbel Distribution

Coefficient of	Nonexceedance	Probability q				
Skewness	0.5	0.8	0.9	0.95	0.98	0.99
γ	Corresponding	Return Period T				
	2	5	10	20	50	100
1.14	−.1692	.7115	1.2999	1.8684	2.6101	3.1704
1.20	−.1709	.7083	1.2977	1.8686	2.6157	3.1817
1.30	−.1761	.6988	1.2913	1.8698	2.6336	3.2172
1.40	−.1846	.6826	1.2798	1.8704	2.6618	3.2753
1.50	−.1905	.6705	1.2705	1.8697	2.6810	3.3166
1.60	−.1949	.6608	1.2628	1.8684	2.6952	3.3483
1.70	−.1998	.6497	1.2536	1.8661	2.7100	3.3827
1.80	−.2048	.6376	1.2431	1.8626	2.7248	3.4186
1.90	−.2097	.6248	1.2316	1.8580	2.7386	3.4542
2.00	−.2143	.6121	1.2197	1.8523	2.7506	3.4876
2.20	−.2217	.5894	1.1972	1.8398	2.7676	3.5418
2.40	−.2264	.5732	1.1804	1.8290	2.7765	3.5761
2.60	−.2291	.5632	1.1697	1.8215	2.7805	3.5954
2.80	−.2309	.5564	1.1622	1.8160	2.7827	3.6080
3.00	−.2326	.5493	1.1543	1.8102	2.7844	3.6202
3.20	−.2348	.5398	1.1435	1.8017	2.7859	3.6355
3.40	−.2376	.5265	1.1281	1.7891	2.7864	3.6548
3.60	−.2408	.5092	1.1073	1.7711	2.7841	3.6760
3.80	−.2439	.4891	1.0822	1.7480	2.7773	3.6950
4.00	−.2464	.4694	1.0565	1.7230	2.7662	3.7076
4.20	−.2478	.4550	1.0372	1.7033	2.7552	3.7127
4.40	−.2481	.4503	1.0308	1.6966	2.7461	3.7136
4.60	−.2477	.4558	1.0383	1.7044	2.7559	3.7125
4.80	−.2467	.4662	1.0523	1.7187	2.7639	3.7090
5.00	−.2461	.4722	1.0603	1.7267	2.7680	3.7062

$$Var(\hat{\theta})=(\theta-x_o)^2\{E_2[1+\Gamma''(2)]-E_3^2\}/N\beta^2D \quad (43b)$$

$$Var(\hat{\beta})=\beta^2E_4/ND \quad (43c)$$

$$Cov(\hat{x}_o,\ \hat{\theta})=(\theta-x_o)^2\{E_5[1+\Gamma''(2)] +E_3\Gamma'(2)\}/N\beta^2D \quad (43d)$$

$$Cov(\hat{\theta},\ \hat{\beta})=-(\theta-x_o)[E_5E_3+E_2\Gamma'(2)]/ND \quad (43e)$$

$$Cov(\hat{x}_o,\ \hat{\beta})=-(\theta-x_o)[E_5\Gamma'(2)+E_3]/ND \quad (43f)$$

and

$$E_1=1+\Gamma''(2)-[\Gamma'(2)]^2=\pi^2/6 \quad (43g)$$

$$E_2 = 1+(1+1/\beta)^2\Gamma(1+2/\beta) - 2(1+1/\beta)\Gamma(1+1/\beta) \quad (43h)$$

$$E_3 = \Gamma(2+1/\beta) + \Gamma'(2+1/\beta) - \Gamma(1+1/\beta) - \Gamma'(2) \quad (43i)$$

$$E_4 = (1+1/\beta)^2[\Gamma(1+2/\beta) - \Gamma^2(1+1/\beta)] = E_2 - E_5^2 \quad (43j)$$

$$E_5 = 1 - \Gamma(2+1/\beta) \quad (43k)$$

$$D = E_1E_4 - [E_3 + \Gamma'(2)E_5]^2 \quad (43l)$$

Note that the expected values of the second derivatives of the log-likelihood function of the log-Gumbel distribution are needed to get variances and covariances of parameters in Eq. (43) and they are given in Appendix B. Finally, the asymptotic variance of estimator of the quantile can be obtained by substituting Eqs. (42) and (43) into (41) as

$$\begin{aligned} S_T^2 = \frac{N^2}{|D|\ (\theta - x_o)^2} & [E_1 + 2S^{-1/\beta}\{-E_1 + E_5 \\ & [1+\Gamma''(2) - \Gamma'(2)\log S] + E_3[\Gamma'(2) - \log S]\} \\ & + S^{-2/\beta}\{E_1 + E_2[1+\Gamma''(2) - 2\Gamma'(2)\log S] \\ & - 2E_5[1+\Gamma''(2) - \Gamma'(2)\log S] - E_3[E_3 + 2\Gamma'(2) \\ & - 2\log S + 2\log SE_5] + E_4(\log S)^2\}] \end{aligned} \quad (44)$$

where $S = -\log(1-1/T)$.

4.3 Standard Error by Probability Weighted Moments

The asymptotic distribution of the sample PWMs can be written as(12,6,13)

$$\begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \sim TVN \left(\begin{matrix} B_0 \\ B_1 \\ B_2 \end{matrix} ; \begin{bmatrix} D_{00}/N & D_{01}/N & D_{02}/N \\ D_{01}/N & D_{11}/N & D_{12}/N \\ D_{02}/N & D_{12}/N & D_{22}/N \end{bmatrix} \right) \quad (45)$$

where $\sim$ reads "is asymptotically distributed as" and TVN is an abbreviation for trivariate normal distribution and Dij are derived in Appendix C as

$$D_{00} = (\theta - x_o)^2[\Gamma(1-2/\beta) - \Gamma^2(1-1/\beta)] \quad (46a)$$

$$D_{01} = [(\theta - x_o)^2/2][2^{2/\beta}\Gamma(1-2/\beta) - (2^{1+1/\beta} - 1)\Gamma^2(1-1/\beta)] \quad (46b)$$

$$D_{02} = [(\theta - x_o)^2/2][\{3^{2/\beta} - 2^{2/\beta}H(1/2)\}\Gamma(1-2/\beta) - 2(3^{1/\beta} - 2^{1/\beta})\Gamma^2(1-1/\beta)] \quad (46c)$$

$$D_{11} = (\theta - x_o)^2 2^{2/\beta}[H(1/2)\Gamma(1-2/\beta) - \Gamma^2(1-1/\beta)] \quad (46d)$$

$$D_{12} = [(\theta - x_o)^2/2][3^{2/\beta}H(1/3)\Gamma(1-2/\beta) - (2\cdot 6^{1/\beta} - 2^{1/\beta})\Gamma^2(1-1/\beta)] \quad (46e)$$

$$D_{22} = (\theta - x_o)^2 3^{2/\beta}[H(2/3)\Gamma(1-2/\beta) - \Gamma^2(1-1/\beta)] \quad (46f)$$

where H(z) is a hypergeometric function. Note that the asymptotic variance-covariances of the sample PWMs of the GEV distribution can be obtained from Eq. (46) by using the relationship of parameters between the log-Gumbel and the GEV distributions.

The asymptotic variance of the estimator of quantile, $\hat{X}_T$ can be found by using the following transformations

$$\begin{matrix} \hat{B}_0 & & \hat{B}_0 & & \hat{B}_0 & & \hat{x}_o & & \\ \hat{B}_1 & \rightarrow & \hat{B}_1 & \rightarrow & \hat{B}_1 & \rightarrow & \hat{\theta} & \rightarrow & \hat{X}_T \\ \hat{B}_2 & & \hat{B}_2 & & \hat{B}_2 & & \hat{\beta} & & \\ & & R & & \hat{b} & & & & \end{matrix}$$

where $R = (3\hat{B}_2 - \hat{B}_0)/(2\hat{B}_1 - \hat{B}_0)$ in the first transformation and $\hat{b}$ is given implicitly by $(1-3^{1/\hat{\beta}})/(1-2^{1/\hat{\beta}}) = R$ in the second transformation. Finally, the asymptotic variance of $\hat{X}_T$ is given by

$$\begin{aligned} S_T^2 = \frac{S^{-2/\beta}}{N} \Big[& (S^{1/\beta} - 1)\mathrm{Var}(\hat{x}_o) + \mathrm{Var}(\hat{\theta}) \\ & + \frac{(\theta - x_o)^2}{\beta^4}\mathrm{Var}(\hat{\beta}) \\ & + 2(S^{1/\beta} - 1)\mathrm{Cov}(\hat{x}_o, \hat{\theta}) + 2(S^{1/\beta} - 1) \\ & \frac{(\theta - x_o)}{\beta^2}\log S\ \mathrm{Cov}(\hat{x}_o, \hat{\beta}) \\ & + \frac{2(\theta - x_o)}{\beta^2}\log S\ \mathrm{Cov}(\hat{\theta}, \hat{\beta}) \Big] \end{aligned} \quad (47)$$

where

$$\mathrm{Var}(\hat{x}_0) = W_o^2D_{00} + 2W_oW_1D_{01} + W_1^2D_{11} + 2W_oW_\beta C_1 H + 2W_1W_\beta C_2H + W_\beta^2CH^2 \quad (48a)$$

$$\mathrm{Var}(\hat{\theta}) = T_o^2D_{00} + 2T_oT_1D_{01} + T_1^2D_{11} + 2T_oT_\beta C_1H + 2T_1T_\beta C_2H + T_\beta^2CH^2 \quad (48b)$$

$$\mathrm{Var}(\hat{\beta}) = CH^2 \quad (48c)$$

$$\mathrm{Cov}(\hat{x}_o, \hat{\theta}) = W_oT_oD_{00} + W_oT_1D_{01} + W_1T_oD_{01} + W_1T_1 D_{11} + W_oT_\beta C_1H + W_1T_\beta C_2H + W_\beta T_oC_1$$

$$H + W_\beta T_1 C_2 H + W_\beta T_\beta C H^2 \quad (48d)$$

$$Cov(\hat{x}_o, \hat{\beta}) = W_o C_1 H + W_1 C_2 H + W_\beta C H^2 \quad (48e)$$

$$Cov(\hat{\theta}, \hat{\beta}) = T_o C_1 H + T_1 C_2 H + T_\beta C H^2 \quad (48f)$$

and

$$W_o = \partial x_o / \partial B_0 = 2^{1/\beta} / (2^{1/\beta} - 1) \quad (49a)$$

$$W_1 = \partial x_o / \partial B_1 = -2/(2^{1/\beta} - 1) \quad (49b)$$

$$W_\beta = \partial x_o / \partial \beta = -(\theta - x_o)\Gamma(1 - 1/\beta)\log(2)2^{1/\beta} / [\beta^2(2^{1/\beta} - 1)] \quad (49c)$$

$$T_0 = \partial\theta/\partial B_0 = [1 - 2^{1/\beta}\Gamma(1 - 1/\beta)]/[(1 - 2^{1/\beta})\Gamma(1 - 1/\beta)] \quad (49d)$$

$$T_1 = \partial\theta/\partial B_1 = -2[1 - \Gamma(1 - 1/\beta)]/[(1 - 2^{1/\beta})\Gamma(1 - 1/\beta)] \quad (49)$$

$$T_\beta = \partial\theta/\partial\beta = \frac{(\theta - x_o)}{\beta^2(1 - 2^{1/\beta})}\{(1 - 2^{1/\beta})\psi(1 - 1/\beta) + [1 - \Gamma(1 - 1/\beta)]\log(2)2^{1/\beta}\} \quad (49f)$$

$$C = [D_{00}(3^{1/\beta} - 2^{1/\beta}) - 2D_{01}(3^{1/\beta} - 1) + 3D_{02}(2^{1/\beta} - 1)]/M \quad (49g)$$

$$C_1 = [D_{01}(3^{1/\beta} - 2^{1/\beta}) - 2D_{11}(3^{1/\beta} - 1) + 3D_{12}(2^{1/\beta} - 1)]/M \quad (49h)$$

$$C_2 = [D_{02}(3^{1/\beta} - 2^{1/\beta}) - 2D_{12}(3^{1/\beta} - 1) + 3D_{22}(2^{1/\beta} - 1)]/M \quad (49i)$$

$$C_3 = [C_1(3^{1/\beta} - 2^{1/\beta}) - 2C_2(3^{1/\beta} - 1) + 3C_3(2^{1/\beta} - 1)]/M \quad (49j)$$

$$M = (\theta - x_o)\Gamma(1 - 1/\beta)(2^{1/\beta} - 1)^2 \quad (49k)$$

$$H = \frac{\beta^2(1 - 2^{1/\beta})^2}{\log(3)\, 3^{1/\beta}(1 - 2^{1/\beta}) - \log(2)\, 2^{1/\beta}(1 - 3^{1/\beta})} \quad (49l)$$

5. Data Application

The annual flood data of the St. Mary's River at Stillwater, Nova Scotia, Canada (1916-1975) are used to illustrate the parameter estimates, quantiles and confidence limits for the log-Gumbel distribution. The chi-square and Kolmogorov-Smirnov goodness of fit tests show that the log-Gumbel model is applicable to the St. Mary's River. The moment estimates (MOM) can be obtained from Eqs. (13), (15) and (16). The maximum likelihood estimates (ML) based on the Newton-Raphson method are obtained from Eq. (22). The probability weighted moments estimates can be found from Eqs. (25) through (27). These parameter estimates are given in Table 2. For given parameter estimates, the quantiles and 95% confidence li-

Table 2. Parameter Estimates of the Log-Gumbel Distribution

Method	Parameter		
	x_o	θ	β
MOM	−2952.481	344.7581	29.91265
ML	−2471.739	345.1803	25.99029
PWM	−2471.733	344.3607	25.64284

Table 3. The Quantiles and 95% Confidence Limits for the Method of Moments

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2	.500	351.8698	385.4070	418.9441
5	.800	454.2984	514.3104	574.3223
10	.900	518.6129	602.3829	686.1528
20	.950	580.3749	688.9659	797.5569
50	.980	661.3514	804.1778	947.0042
100	.990	723.0700	892.8958	1062.7220
500	.998	869.6490	1106.0090	1342.3690

Table 4. The Quantiles and 95% Confidence Limits for the Method of Maximum Likelihood

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2	.500	350.5921	385.1857	419.7792
5	.800	460.0687	512.5317	564.9947
10	.900	525.3752	599.9536	674.5320
20	.950	578.3453	686.2164	794.0874
50	.980	630.6382	801.4820	972.3257
100	.990	657.1479	890.6065	1124.0650
500	.998	674.3317	1105.9550	1537.5790

Table 5. The Quantiles and 95% Confidence Limits for the Method of Probability Weighted Moments

Return Period T	Nonexceedance Probability q	Lower Limit	Quantile $\hat{X}_T$	Upper Limit
2	.500	349.2581	384.9001	420.5421
5	.800	460.2496	513.9969	567.7442
10	.900	526.8259	602.6641	678.5023
20	.950	580.0417	690.1891	800.3365
50	.980	630.5775	807.1919	983.8063
100	.990	654.1571	897.6979	1141.2390
500	.998	659.1465	1116.4670	1573.8880

mits, corresponding to return periods T=2, 5, 10, 20, 50, 100 and 500 of the log-Gumbel distribution are evaluated in Table 3, 4 and 5, respectively. The confidence limits of the PWM are close to the ML and show the most tolerable for the large return periods. The confidence limits of the MOM show the most restricted limits for the large return periods.

As an alternative, the frequency factors can be used to estimate the quantiles. For example, the sample mean, standard deviation and skewness coefficient of the St. Mary's River flood data are 412.145, 148.01, 1.3557, respectively. From Table 1, the value of the frequency factor for return period 100 year is 3.2496 and then the quantile for return period 100 year can be $412.145+3.2496 \times 148.01=893.118$.

6. Conclusions

The log-Gumbel distribution in real space is defined by transforming the conventional log-Gumbel distribution in log space. The parameter estimation techniques, such as the methods of moment (MOM), maximum likelihood (ML) and probability weighted moments (PWM), are proposed to estimate the parameters for this model. The results of the parameter estimations are as follows: An iterative procedure such as the Newton-Raphson is needed for each method to estimate parameters for the three parameter log-Gumbel model. For the two parameter log-Gumbel model, the PWM method does not need any iterative procedure. Hence, the PWM estimates of the parameters can be used as the initial values for the other methods even for the three parameter log-Gumbel model.

The asymptotic variances of the MOM, ML and PWM quantile estimators for the three parameter log-Gumbel distribution are derived as a function of sample size, return period and parameters as a major contribution in this study. All formulae do have very long and complicated forms. However, they can be determined numerically.

The confidence limits of the quantiles based on the MOM, ML and PWM methods can be obtained by using the corresponding asymptotic variances. Model applications to the annual flood data of the St. Mary's River show that the confidence limits for the MOM method are most restricted for the large return periods and the confidence limits for the ML and PWM are close to each other.

The derived asymptotic variances of the quantile estimators are based on the three parameter Weibull model. To apply the confidence limits of the quantiles for the two parameter Weibull distribution, the derivation of the asymptotic variances for the two parameter model is very useful for a future study because the parameter estimation based on the three parameter Weibull model may not be applicable for some real flood data.

References

1. National Environmental Research Council, *Flood Studies Report*, Vol. I, London, United Kingdom, 1975.
2. Shen, H.W., Bryson, M.C. and Ochoa, I.D., "Effect of Tail Behavior Assumptions on Flood Predictions", *Water Resources Research*, Vol. 16, No. 2, 1980, pp. 361-364.
3. Ochoa, I.D., Bryson, M.C. and Shen, H.W., "On the Occurrence and Importance of Parentian-Tailed Distributions in Hydrology", *Journal of Hydrology*, Vol. 48, 1980, pp. 53-62.
4. Prescott, P. and Walden, A.T., "Maximum Likeli-

hood Estimation of the Parameters of the Generalized Extreme-Value Distribution", *Biometrika*, Vol. 67 No. 3, 1980, pp. 723-724.

5. Prescott, P. and Walden, A.T., "Maximum Likelihood Estimation of the Three-Parameter Generalized Extreme-Value Distribution from Censored Samples", *Journal of Statistical Computational Simulation*, Vol. 16, 1983, pp. 241-250.
6. Hosking, J.R.M., Wallis, J.R. and Wood, E.F., "Estimation of the Generalized Extreme-Value Distribution by the Method of Probability Weighted Moments", *Technometrics*, Vol. 27 No. 3, 1985, pp. 246-261.
7. Hosking, J.R.M., *The Theory of Probability* Weighted Moments, Research Report RC12210, IBM TJ. Watson Research Center, Yorktown Heights, New York, 1986.
8. Liu, L.H. and Stedinger, J.R., "Variance of Two- and Three-Parameter GEV/PWM Quantile Estimators: Formulae, Confidence Intervals, and a Comparison", *Journal of Hydrology*, Vol. 138, 1992, pp. 247-267.
9. Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R., "Probability Weighted Moments: Definition and Relation to Parameters of Several Distributions Expressible in Inverse Form", *Water Resources Research*, Vol. 15 No. 5, 1979, pp. 1049-1954.
10. Chow, V.T., "A General Formula for Hydrologic Frequency Analysis", *Trans. American Geophysical Union*, Vol. 32, 1946, pp. 231-237.
11. Kite, G.W., *Frequency and Risk Analyses in Hydrology*, Water Resources Publications, Fort Collins, Colorado, 1977.
12. Chernoff, H., Gastwirth, J.L. and Johns, M.V., "Asymptotic Distribution of Linear Combinations of Functions of Order Statistics with Applications to Estimation", *Annals of Mathematical Statistics*, Vol. 38, 1967, pp. 52-72.
13. Heo, J.H., Boes, D.C. and Salas, J.D., *Regional Flood Frequency Modeling* and *Estimation*, Water Resources Papers, No. 101, Colorado State Univ., Fort Collins, Colorado, 1990.

(接受 : 1993. 6. 23)

Appendix A Second Partial Derivatives of the Log-Likelihood Function of the Log-Gumbel Distribution.

$$-\partial^2 LL/\partial x_o^2 = N\beta/(\theta-x_o) - (\beta+1)\Sigma(x_i-x_o)^{-2} + \beta(\beta-1)/(\theta-x_o)^{\beta-2}\Sigma(x_i-x_o)^{-\beta} - 2\beta^2(\theta-x_o)^{\beta-1}\Sigma(x_i-x_o) \tag{A1}$$

$$-\partial^2 LL/\partial\theta\partial x_o = -N\beta/(\theta-x_o)^2 - \beta(\beta-1)(\theta-x_o)^{\beta-2}\Sigma(x_i-x_o)^{-\beta} + \beta^2(\theta-x_o)^{\beta-1}\Sigma(x_i-x_o)^{-\beta-1} \tag{A2}$$

$$-\partial^2 LL/\partial x_o\partial\beta = N/(\theta-x_o) - \Sigma(x_i-x_o)^{-1} + \Sigma\left[\frac{\theta-x_o}{x_i-x_o}\right]^{\beta}\left\{1+\beta\log\left[\frac{\theta-x_o}{x_i-x_o}\right]\right\} \times\left[\frac{1}{x_i-x_o}-\frac{1}{\theta-x_o}\right] \tag{A3}$$

$$-\partial^2 LL/\partial\theta^2 = N\beta/(\theta-x_o) + \beta(\beta-1)(\theta-x_o)^{\beta-2}\Sigma(x_i-x_o)^{-\beta} \tag{A4}$$

$$-\partial^2 LL/\partial\theta\partial\beta = N/(\theta-x_o) + \Sigma\left[\frac{\theta-x_o}{x_i-x_o}\right]^{\beta} \left\{1+\beta\log\left[\frac{\theta-x_o}{x_i-x_o}\right]\right\}\cdot\left[\frac{1}{\theta-x_o}\right] \tag{A5}$$

$$-\partial^2 LL/\partial\beta^2 = -N/\beta^2 - \Sigma\left[\frac{\theta-x_o}{x_i-x_o}\right]^{\beta}\left\{\log\left[\frac{\theta-x_o}{x_i-x_o}\right]\right\}^2 \tag{A6}$$

where $\Sigma = \sum_{i+1}^{N}$

Appendix B Expected Values of the Second Partial Derivatives of the Log-Likelihood Function of the Log-Gumbel Distribution.

$$E\left[-\frac{\partial^2 LL}{\partial x_o^2}\right] = \frac{N\beta^2}{(\theta-x_o)^2}[1+(1+1/\beta)^2\Gamma(1+2/\beta) - 2(1+1/\beta)\Gamma(1+1/\beta)] \tag{B1}$$

$$E\left[-\frac{\partial^2 LL}{\partial\theta^2}\right] = \frac{N\beta^2}{(\theta-x_o)^2} \tag{B2}$$

$$E\left[-\frac{\partial^2 LL}{\partial\beta^2}\right] = \frac{N}{\beta^2}[1+\Gamma''(2)] \tag{B3}$$

$$E\left[-\frac{\partial^2 LL}{\partial x_o\partial\theta}\right] = -\frac{N\beta^2}{(\theta-x_o)^2}[1-\Gamma(2+1/\beta)] \tag{B4}$$

$$E\left[-\frac{\partial^2 LL}{\partial x_o\partial\beta}\right] = \frac{N^2}{(\theta-x_o)}[\Gamma(2+1/\beta)$$

$$+\Gamma'(2+1/\beta)-\Gamma(1+1/\beta)-\Gamma'(2)] \quad \text{(B5)}$$

$$E\left[-\frac{\partial^2 LL}{\partial\theta\partial\beta}\right]=\frac{N}{(\theta-x_o)}\Gamma'(2) \quad \text{(B6)}$$

where $\Gamma(w)$ is a gamma function with argument w and $\Gamma'(w)$ and $\Gamma''(w)$ are the first and second partial derivatives of a gamma function, respectively.

Appendix C Derivation of Elements Dij of Matrix D for the Probability Weighted Moments Br

Let $B_r=E[X\{F(x)\}^r]$, $r=0, 1,\cdots, m$, denote population probability weighted moments (PWM) and $\hat{B}_r$ be the corresponding sample PWM. As N goes to infinity, $N^{1/2}(B_r-\hat{B}_r)$, $r=0, 1,\cdots, m-1$, converges in distribution to the trivariate normal distribution N(0, D)[7]. The elements of D_{ij} of matrix D are given by

$$D_{ij}=J_{ij}+J_{ji} \quad \text{(C1)}$$

and

$$J_{ij}=\iint_{x<y}\{F(x)\}^i\{F(y)\}^j F(x)[1-F(y)]dxdy \quad \text{(C2)}$$

Therefore

$$D_{00}=2\iint_{x<y}F(x)[1-F(y)]dxdy=(\theta-x_o)^2[\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)] \quad \text{(C3)}$$

$$D_{01}=\iint_{x<y}[F(x)+F(y)]F(x)[1-F(y)]dxdy =[(\theta-x_o)^2/2][2^{2/\beta}\Gamma(1-2/\beta)-(2^{1+1/\beta}-1)\Gamma^2(1-1/\beta)] \quad \text{(C4)}$$

$$D_{02}=\iint_{x<y}[\{F(x)\}^2+\{F(y)\}^2]F(x)[1-F(y)]dxdy =[(\theta-x_o)^2/2][\{3^{2/\beta}-2^{2/\beta}H(1/2)\}\Gamma(1-2/\beta) -2(3^{1/\beta}-2^{1/\beta})\Gamma^2(1-1/\beta)] \quad \text{(C5)}$$

$$D_{11}=2\iint_{x<y}F(x)F(y)F(x)[1-F(y)]dxdy =(\theta-x_o)^2 2^{2/\beta}[H(1/2)\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)] \quad \text{(C6)}$$

$$D_{12}=\iint_{x<y}[\{F(x)\}^2F(y)+F(x)\{F(y)\}^2]F(x)[1-F(y)]dxdy=[(\theta-x_o)^2/2][3^{2/\beta}H(1/3)\Gamma(1-2/\beta)-(2\cdot 6^{1/\beta}-2^{1/\beta})\Gamma^2(1-1/\beta)] \quad \text{(C7)}$$

$$D_{22}=2\iint_{x<y}\{F(x)\}^2\{F(y)\}^2F(x)[1-F(y)]dxdy =(\theta-x_o)^2 3^{2/\beta}[H(2/3)\Gamma(1-2/\beta)-\Gamma^2(1-1/\beta)] \quad \text{(C9)}$$

where $H(z)=F(-2/\beta, -1/\beta; 1-1/\beta; -z)$ is a hypergeometric function.