ON REGULARITY AND APPROXIMATION FOR HAMILTON–JACOBI EQUATIONS

BUM IL HONG

1. Introduction

Discontinuities may form in the derivative with respect to x of the solution u(x, t) of the Hamilton-Jacobi equation

\begin{align*}
  u_t + f(\nabla u) &= 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{align*}

(H-J)

even though the flux f and the initial data u_0 are smooth and u_0 has compact support. The difficulties in defining the correct solutions of (H-J) were overcome by Crandall and Lions [2], who introduced the notion of viscosity solutions. They also showed that viscosity solutions of (H-J) are stable in $L^\infty$ with respect to perturbation in the initial data. Consequently the space of Lipschitz continuous functions forms a regularity space for (H-J). In one space dimension, Hong [6] has recently shown that if f is strictly convex, has three bounded derivatives, and $u'_0 \in BV(\mathbb{R})$, then for a finite interval $I$, $0 < \alpha < 3$ and $q \in (0, \infty]$, $A^\alpha_q(C(I))$ are also regularity spaces for (H-J). As a corollary, since the Besov spaces $B^{\alpha-1}_q(L^q)$ for $1 < \alpha < 3$ and $q = 1/\alpha$ are equivalent to $A^\alpha_q(C(I))$, it follows that whenever $u'_0$ is in $B^{\alpha-1}_q(L^q)$, $u_x(\cdot, t)$ remains in the same space for all positive time. We will prove this fact directly.

It may be useful to sketch the work by Hong [6]. He first showed that solutions of (H-J) are stable under perturbations in the nonlinearity f as well as the initial data u_0. In particular, if $u(x, t)$ solves (H-J) and $v(x, t)$ solves the similar problem

\begin{align*}
  v_t + g(\nabla v) &= 0, \quad x \in \mathbb{R}^n, \quad t > 0,
\end{align*}

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.
with initial data \( v(x,0) = v_0(x) \), then for positive \( t \),
(\text{H}) \quad \|u(\cdot,t) - v(\cdot,t)\|_{L^\infty(R^n)} \leq \|u_0 - v_0\|_{L^\infty(R^n)} + t\|f - g\|_{L^\infty(R^n)}.

A specific construction in one space dimension is then made in which \( f \) is approximated in \( L^\infty(R) \) to order \((2^n)^{-3}\) by a \( C^2 \) piecewise quadratic function \( f_n \) and \( u_0 \) is approximated in \( L^\infty \) by a continuous, piecewise quadratic polynomial \( v_0 \) with \( 2^n \) free knots. It is then shown that the solution of \( v(\cdot,t) \) of
\[
v_t + f_n(v_x) = 0, \quad x \in R, \quad t > 0,
\]
\[
v(x,0) = v_0(x), \quad x \in R,
\]
is again continuous, piecewise quadratic for all time and has no more than \( C2^n \) pieces where for some \( C \). The Hong's stability theorem \( \text{H} \) shows that \( u(\cdot,t) \) can be approximated with an error not exceeding the error of approximation of \( u_0 \) plus \( O((2^n)^{-3}) \). The regularity theorem is then proved by using the characterization developed by DeVore and Popov [4], using the result of Petrushev [11].

This approach does not carry over directly to the spaces \( B^{q-1}_q \) because one must deal with \( u_x(\cdot,t) \). To overcome this difficulty, we use the relationship between the equations of \( \text{H-J} \) and hyperbolic conservation laws. This relationship is very simple: if \( u \) is the viscosity solution of \( \text{H-J} \), then \( v = u_x \) is the entropy solution of scalar conservation laws
\[
v_t + f(v)_x = 0, \quad x \in R, \quad t > 0,
\]
(\text{C}) \quad v(x,0) = v_0(x) = u'_0(x), \quad x \in R.

Moreover, Lucier [9] showed a similar stability result like \( \text{H} \): if \( v \) solves \( \text{C} \) and \( w \) solves the similar problem
\[
w_t + g(w)_x = 0, \quad x \in R, \quad t > 0,
\]
\[
w(x,0) = w_0(x), \quad x \in R,
\]
then for positive \( t \),
(\text{L}) \quad \|v(\cdot,t) - w(\cdot,t)\|_{L^1(R)} \leq \|v_0 - w_0\|_{L^1(R)}
\quad + t\|f' - g'\|_{L^\infty(R)} \min\{\|u_0\|_{BV(R)}, \|w_0\|_{BV(R)}\}

Using the relationship between \( \text{H-J} \) and \( \text{C} \), and the stability result \( \text{L} \) given by Lucier, we prove the following theorem that is the main result of this paper.
THEOREM 1.1. Let $\alpha \in (1, 3)$. Suppose that $u_0$ is Lipschitz continuous, has support in $[0,1]$, and that $u'_0 \in BV(\mathbb{R})$. Let $f(0) = 0$. Suppose that $f'' > 0$ and that $f'$ and $f'''$ are bounded in $\Omega = \{y||y| \leq C|u'_0|_{BV(\mathbb{R})}\}$, where $C$ will be specified later. If $u'_0$ is in $B^q_{\alpha-1}(L^q([0,1]))$, where $q = 1/\alpha$, then $u_\cdot (\cdot, t)$ has support in $I_t = [t \inf_{\rho \in \Omega} f(\rho), 1 + t \sup_{\rho \in \Omega} f(\rho)]$ and $u_\cdot (\cdot, t) \in B^q_{\alpha-1}(L^q(I_t))$.

2. Preliminaries

Analysis by the method of characteristics shows that $C^1$ solutions of (C) are constant along the characteristic lines $x = x_0 + tf'(v_0(x_0))$, so near the line $t = 0$ the solutions of (C) are of the form

$$v(x, t) = v_0(x - f'(v)t).$$

Since discontinuities may develop in $v$, the implicit equation (2.1) may no longer be solution. However, the minimization property introduced by Lax [8] makes it possible for us to find local solutions of (2.1), at least when $f$ is convex. He picks out a specific value $y := y(x, t)$ among many possible solutions of $\frac{x-y}{t} = f'(v_0(y))$ so that $v(x, t) = v_0(y)$. He also showed that $y(x, t)$ is an increasing function of $x$ for fixed $t$. Moreover, if $y(x, t)$ is discontinuous at $x$, then shock occurs.

Let $I$ be a finite interval. We are going to use the following notations

$$\|f\|_{L^p(I)} := \left(\frac{1}{|I|} \int_I |f(x)|^p \, dx\right)^{1/p},$$

if $0 < p < \infty$ and

$$\|f\|_{L^\infty(I)} := \sup_{x \in I} |f(x)|.$$

The following inequalities are well known for polynomials $P$ of degree no more than $k$; see DeVore and Sharley [5].

For each $k = 0, 1, \ldots$ and $p$, $q \in (0, \infty]$ there is a constant $C$ such that for all polynomials $P$ of degree $\leq k$,

$$\|P\|_{L^p(I)} \leq C\|P\|_{L^q(I)}.$$
For each \( k = 0, 1, \ldots \) and \( p \in (0, \infty] \) there is a constant \( C \) such that for all polynomials \( P \) of degree \( \leq k \),

\[
\|P'\|_{L^p(I)}^* \leq C |I|^{-1} \|P\|_{L^p(I)}^*.
\]

Consider the functions in \( L^1(I) \) and a finite interval \( I \). For any \( f \in L^1(I) \) and any positive integer \( n \), let \( E_n^2(f) := \inf \|f - \phi\|_{L^1(I)} \) where the infimum taken over all piecewise polynomial functions \( \phi \) defined on \( I \) of degree less than 2 with at most \( 2^n - 1 \) free interior knots (i.e., \( 2^n \) polynomial pieces). For \( 1 < \alpha < 3 \) and \( q = \frac{1}{\alpha} \), define \( A^{\alpha-1}_q(L^1(I)) \) to be the set of functions for which

\[
\|f\|_{A^{\alpha-1}_q(L^1(I))} := \|f\|_{L^1(I)} + \left( \sum_{n=0}^{\infty} [2^n(\alpha-1)E_n^2(f)]^q \right)^{1/q} < \infty.
\]

We now define Besov spaces. For \( \alpha \in (0, \infty) \), \( q \in (0, \infty] \) and \( p \in (0, \infty] \), the Besov spaces \( B^\alpha_q(L^p(I)) \) are defined as follows: Pick any integer \( r > \alpha \), let \( \Delta_h f(x) \) be the forward difference of \( f \) at \( x \) with step size \( h \) (i.e., \( \Delta_h^0 f(x) := f(x) \) and \( \Delta_h f(x) := \Delta_h^{r-1} f(x + h) - \Delta_h^{r-1} f(x) \)). Let \( I_h = \{x \in I \mid x + rh \in I \} \). Define \( \omega_r(f, t)_p = \sup_{|k| < t} \|\Delta_h^k f\|_{L^p(I_h)} \). The Besov space \( B^\alpha_q(L^p(I)) \) is defined to be the set functions \( f \) for which

\[
\|f\|_{B^\alpha_q(L^p(I))} := \|f\|_{L^p(I)} + \left( \int_0^\infty [t^{-\alpha} \omega_r(f, t)_p]^q \, dt \right)^{1/q} < \infty.
\]

We are particularly interested in the spaces \( B^{\alpha-1}_q(I) := B^{\alpha-1}_q(L^q(I)), \)

\( 1 < \alpha < 3 \), where \( q = 1/\alpha \). Based on work by Petrushev [11], DeVore and Popov [4] showed the following theorem.

**Theorem 2.1.** If \( 1 < \alpha < 3 \) and \( q = 1/\alpha \), then

\[
B^{\alpha-1}_q(L^q(I)) = A^{\alpha-1}_q(L^1(I)).
\]
3. Regularity for Hamilton-Jacobi equations

In this section we prove Theorem 1.1. The proof is divided into several steps; we first construct a certain approximation to the solution $v(x, t)$ of (C).

The initial data $v_0 = u'_0$ has support in $I = [0, 1]$ since we assume that $u_0$ has support in $I$. We now fix $r = 2$. Assume that $P_n$ is the best $L^1(I)$ approximation to $v_0$ having support in $I$ if $||v_0 - P_n||_{L^1(I)} = E^2_n(v_0)$. Let $\{\tau_i\}_{i=0}^n$, with $\tau_0 = 0$ and $\tau_n = 1$, be the ordered set of knots $P_n$. Define each interval $I_i$ by $I_i := [\tau_i, \tau_{i+1}]$ with $|I_i| := \tau_{i+1} - \tau_i$.

**Lemma 3.1.** If $P_n$ is the best $L^1(I)$ approximation to $v_0$, then

$$|P_n|_{BV(R)} \leq |v_0|_{BV(R)} = |u'_0|_{BV(R)}.$$

**Proof.** Assume that $v_0$ is right continuous. For each $i$ let $C_i$ be the number satisfying $C_i := v_0(x_i)$ for some $x_i \in I_i$. Suppose that $C(x)$ is the piecewise constant function taking the value $C_i$ on each $I_i$. Then

$$|C(x)|_{BV(R)} \leq |v_0|_{BV(R)}.$$  

Therefore

$$|P_n|_{BV(I_i)} = \int_{I_i} |P'_n| \, dx$$

$$= \int_{I_i} |P'_n - C'_i| \, dx$$

$$\leq \frac{C}{|I_i|} \int_{I_i} |P_n - C_i| \, dx$$

$$\leq \frac{C}{|I_i|} \int_{I_i} \{ |P_n - v_0| + |v_0 - C_i| \} \, dx$$

$$\leq \frac{C}{|I_i|} \int_{I_i} |v_0 - C_i| \, dx$$

$$\leq \frac{C}{|I_i|} \sup_{x \in I_i} \int_{\tau_i}^{x} |x - s| |v'_0(s)| \, ds$$

$$\leq C \int_{I_i} |v'_0| \, dx$$

$$\leq C |v_0|_{BV(I_i)}.$$
Here the first inequality is (2.3).

We now measure the jump $|P_n(\tau_1^+) - P_n(\tau_1^-)|$. It is clear that

$$|P_n(\tau_1^+) - P_n(\tau_1^-)| \leq \|P_n(x) - C_i\|_{L^\infty(I_i)} + \|P_n(x) - C_{i-1}\|_{L^\infty(I_{i-1})} + |C_i - C_{i+1}|,$$

and

$$\|P_n(x) - C_i\|_{L^\infty(I_i)} \leq \frac{C}{|I_i|} \int_{I_i} |P_n(x) - C_i| \, dx$$

$$\leq \frac{C}{|I_i|} \int_{I_i} |v_0 - C_i| \, dx$$

$$\leq C|v_0|_{BV(I_i)}.$$

The second conclusion is (3.1). Hence the jump is not larger than

$$C|v_0|_{BV(I_i)} + C|v_0|_{BV(I_{i+1})} + |C_{i+1} - C_i|.$$

So we get

$$|P_n|_{BV(R)} \leq C(|v_0|_{BV(R)} + |C(x)|_{BV(R)})$$

$$\leq C|v_0|_{BV(R)}.$$

This completes the proof. \(\Box\)

This $C$ is the constant of Theorem 1.1. Moreover one can easily see that

$$\|P_n\|_{L^\infty(R)} \leq \|v_0\|_{L^\infty(R)} = \|u'_0\|_{L^\infty(R)}.$$

So, the ranges of $P_n$ and $v_0$ are contained in $\Omega = \{y \mid |y| \leq C|u'_0|_{BV(R)}\}$.

We now construct an approximation $f_n$ to $f$ on $\Omega$. There is a $C^1$ piecewise quadratic approximation $f_n$ to $f$ with knots at the points $j/2^n$, $j \in Z$, that is defined by $f'_n(j/2^n) = f'(j/2^n)$ with $f_n(x)$ is a linear function between $j/2^n$ and $j + 1/2^n$ and $f_n(0) = f(0)$. Moreover $\|f'_n - f''\|_{L^\infty(R)} \leq C\|f'''\|_{L^\infty(R)}(\frac{1}{2^n})^2$; see [1] and [3]. Consider now the perturbed problem

$$(P_n)_t + f_n(P_n)_x = 0, \quad x \in R, \quad t > 0,$$

$$P_n(x,0) = P_n(x), \quad x \in R.$$
It is shown in [9] and [10] that the solution $P_n(\cdot, t)$ is again piecewise linear for each time $t > 0$ and the number of knots is no more than $C2^n$ where $C$ depends only on $t$, $|v_0|_{BV(R)}$ and the degree 1. The stability theorem (L) then shows that

$$\|v(\cdot, t) - P_n(\cdot, t)\|_{L^1(R)} \leq \|v_0 - P_n(\cdot, 0)\|_{L^1(R)} + Ct\|f' - f_n'\|_{L^\infty(R)}|v_0|_{BV(R)}$$

(3.2)

$$\leq \|v_0 - P_n(\cdot, 0)\|_{L^1(R)} + Ct\|f'''\|_{L^\infty(R)} \frac{1}{(2^n)^2}.$$ 

Proof of Theorem 1.1. The first conclusion is classical. We will prove the second conclusion. We obtain from (3.2) that $P_n(\cdot, t) \to v(\cdot, t)$ in $L^1(I_t)$ and so we write

$$u(\cdot, t) = \sum_{n=-1}^{\infty} (P_{n+1}(\cdot, t) - P_n(\cdot, t)),$$

where $P_{-1}(\cdot, t) = 0$. We now count some points with respect to $x$. The first are the points of intersection of $P_{n+1}(x, t)$ and $P_n(x, t)$ and the second type of points are points where either $P_{n+1}(x, t)$ or $P_n(x, t)$ is discontinuous in $x$. The total number of points of these two types is clearly no more than $C2^n$. One can therefore see that $P_{n+1}(x, t) - P_n(x, t)$ is a discontinuous, piecewise linear function in $x$ with at most $C2^n$ pieces. Let each interval $I_i$ be determined by two adjacent points from type one and type two. Let $P_{n+1}(x, t) - P_n(x, t) = \sum_{i=1}^{K} L_i$, $K \leq C2^n$, where each linear polynomial piece $L_i$, in $x$ at fixed time $t$, is defined on $I_i$ and vanishes outside $I_i$.

We will use a generic constant $C$ depending on the constant $C$ in (3.2) and $\|f(3)\|_{L^\infty(R)}$. We fix $h$. For each $i$ define three subsets $A_i$, $B_i$ and $C_i$ of $I_i$ by $A_i = \{x \in I_i | x, x + h, x + 2h \in I_i\}$, $B_i = \{x \in I_i | x \notin A_i$ and $\{x, x + h, x + 2h\} \cap I_i \neq \emptyset\}$ and $C_i = I_i - (A_i \cup B_i)$. Then $A_i = \emptyset$ if $h > \frac{|I_i|}{2}$ and $B_i$ has measure no more than $4\min(h, |I_i|)$.

For $x \in B_i$, we fix $p > 1$ with $p < \frac{4}{3}$. Since

$$|\Delta_h^2(L_i, x)| \leq |L_i(x + 2h) - 2L_i(x + h) + L_i(x)|$$

$$\leq 2|L_i(x + 2h) + L_i(x + h) + L_i(x)|$$

$$\leq C|L_i(x)|,$$
by Hölder inequality, we have

\[
\int_{B_i} |\Delta_h^2(L_i, x)|^q \, dx \leq |B_i|^{1 - q/p} \left( \int_{I_i} |L_i(x)|^p \, dx \right)^{q/p}
\]

\[
\leq C \left( \min(h, |I_i|) \right)^{1 - q/p} \left( \int_{I_i} |L_i(x)|^p \, dx \right)^{q/p}
\]

\[
= C \min(h, |I_i|)^{1 - q/p} |I_i|^{q/p} \left( \|L_i\|_{L^p(I_i)}^* \right)^q
\]

\[
\leq C \min(h, |I_i|)^{1 - q/p} |I_i|^{q/p} \left( \|L_i\|_{L^1(I_i)}^* + \frac{1}{(2^n)^2} \right)^q
\]

(3.3)

\[
= C \min(h, |I_i|)^{1 - q/p} |I_i|^{-q + q/p} \left( \int_{I_i} |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q.
\]

Let \( x \in C_i \). Since \( \Delta_h^2(L_i, x) = 0 \),

(3.4)

\[
\int_{C_i} |\Delta_h^2(L_i, x)|^q \, dx = 0.
\]

We finally consider \( x \in A_i \). For each \( x \in I_i \) there is \( \xi \in I_i \) such that

\[
|\Delta_h^2(L_i, x)| = C h |L_i'(\xi)|,
\]

because \( L_i \) is linear on \( I_i \). Hence,

\[
\int_{A_i} |\Delta_h^2(L_i, x)|^q \, dx \leq C h^q \int_{I_i} |L_i'(x)|^q \, dx
\]

\[
\leq C h^q |I_i|^{-q} \int_{I_i} |L_i(x)|^q \, dx
\]

\[
\leq C h^q |I_i|^{-q} (\|L_i\|_{L^1(I_i)}^*)^q
\]

\[
\leq C h^q |I_i|^{-q} \left( \int_{I_i} |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q.
\]

Here the second inequality is (2.3) and the third inequality is (2.2).
(3.3), (3.4) and (3.5) now yield that
\[
\int_R |\Delta h^2(L_i, x)|^q \, dx \leq C \left( h^q |I_i|^{1-2q} \chi(h) + [\min(h, |I_i|)]^{1-q/p} |I_i|^{q/p-q} \right) \\
\times \left( \int |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q,
\]
(3.6)
where \( \chi(h) \) is the characteristic function on \([0, |I_i|/h]\).

Because one can easily show that \( \omega_2(L_i, h)^q \) is less than the right hand side of the inequality (3.6),
\[
\int_0^{\infty} [h^{-(\alpha-1)} \omega_2(L_i, h)^q] \, dh/h \\
= \int_0^{\infty} h^{-q(\alpha-1)} \omega_2(L_i, h)^q \, dh/h \\
\leq C \left( \int |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q \left( \int_0^{\infty} h^{-q(\alpha-1)-1} \frac{|I_i|^{1-2q} \chi(h)}{h^{-(\alpha-2)-1}} \, dh \\
+ \int_0^{\infty} h^{-q(\alpha-1)-1} [\min(h, |I_i|)]^{1-q/p} |I_i|^{-q+q/p} \, dh \right) \\
\leq C \left( \int |I_i| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q \left( |I_i|^{1-2q} \int_0^{\infty} h^{-q(\alpha-2)-1} \, dh \\
+ |I_i|^{-q} \int_{|I_i|}^{\infty} h^{-q(\alpha-1)-1} \, dh + |I_i|^{-q+q/p} \int_0^{\infty} h^{-q(\alpha-1)-1} h^{-q/p} \, dh \right) \\
\leq C |I_i|^{1-q} \left( \int |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q \\
= C \left( \int |L_i(x)| \, dx + \frac{|I_i|}{(2^n)^2} \right)^q,
\]
because \( 1 - q\alpha = 0 \).

Now, since \( q < 1 \), we have
\[
\omega_2(P_{n+1} - P_n, h)^q \leq \sum_{i=1}^{K} \omega_2(L_i, h)^q.
\]
(3.8)
It follows immediately that

\[ \int_0^\infty h^{-(\alpha-1)q} \omega_2(P_{n+1} - P_n, h)^q dh/h \]

\[ \leq \sum_{i=1}^K \int_0^\infty h^{-(\alpha-1)q} \omega_2(L_i, h)^q dh/h \]

\[ \leq C \sum_{i=1}^K \left( \int_{I_i} |L_i(x)| dx + \frac{|I_i|}{(2^n)^2} \right)^q \]

(3.9)

\[ \leq CK^{1-q} \left( \|P_{n+1}(\cdot, t) - P_n(\cdot, t)\|_{L^1(I_i)} + \frac{|I_i|^q}{(2^n)^2} \right) \]

\[ \leq CK^q(\alpha-1) \left( \|P_{n+1}(\cdot, t) - P_n(\cdot, t)\|_{L^1(I_i)} + \frac{|I_i|^q}{(2^n)^2} \right) \]

\[ \leq C2^{q(\alpha-1)n} \left( \|P_{n+1}(\cdot, t) - P_n(\cdot, t)\|_{L^1(I_i)} + \frac{|I_i|^q}{(2^n)^2} \right). \]

Here the first inequality is (3.8), the second is (3.7), the third is Hölder inequality and the fourth follows from the facts that \( 1 - q = q(\alpha - 1) \) and \( 0 < q < 1 \).

Using (3.8), (3.9) and Theorem 2.1, we have

\[ \int_0^\infty [h^{-(q-1)} \omega_2(u_x, h)]^q dh/h \]

\[ = \int_0^\infty [h^{-(q-1)} \omega_2(u, h)]^q dh/h \]

\[ \leq \sum_{n=-1}^\infty \int_0^\infty [h^{-(q-1)} \omega_2(P_{n+1} - P_n, h)]^q dh/h \]

\[ \leq C \sum_{n=-1}^\infty 2^{nq(\alpha-1)} \left( \|P_{n+1}(\cdot, t) - P_n(\cdot, t)\|_{L^1(I_i)} + 2^{-2nq} \right) \]

\[ \leq C \sum_{n=-1}^\infty 2^{nq(\alpha-1)} \left( \|P_n(\cdot, 0) - v_0(\cdot, 0)\|_{L^1(I)} + 2^{-n2q} + 2^{-2nq} \right) \]
On regularity and approximation for Hamilton-Jacobi equations

\[
\begin{align*}
&\leq C \sum_{i=-1}^{\infty} 2^{nq(\alpha-1)} \left( E_n^2(v_0)^q + 2^{-2nq} \right) \\
&= C \sum_{i=-1}^{\infty} 2^{nq(\alpha-1)} \left( E_n^2(u_0')^q + 2^{-2nq} \right) \\
&\leq C \|u_0'\|_{B_{q^{-1}}^\infty(L^1([0,1]))}^q + C,
\end{align*}
\]

because from (3.2), for \( n = -1, 0, ..., \)

\[
\begin{align*}
&\|P_{n+1}(\cdot, t) - P_n(\cdot, t)\|_{L^1(I_t)}^q \\
&\leq \left( \|P_{n+1}(\cdot, t) - v_0(\cdot, t)\|_{L^1(I_t)} + \|P_n(\cdot, t) - v_0(\cdot, t)\|_{L^1(I_t)} \right)^q \\
&\leq \left( \|P_{n+1}(\cdot, 0) - v_0(\cdot, 0)\|_{L^1(I)} + \|P_n(\cdot, 0) - v_0(\cdot, 0)\|_{L^1(I)} + \frac{1}{(2^n)^2} \right)^q \\
&\leq C \left( \|P_n(\cdot, 0) - v_0(\cdot, 0)\|_{L^1(I)}^q + 2^{-2nq} \right)
\end{align*}
\]

By (3.10),

\[
\|u(\cdot, t)\|_{B_{q^{-1}}^\infty(L^1(I_t))} \leq C \|u_0'\|_{B_{q^{-1}}^\infty(L^1(I))} + C.
\]

This completes the proof. \( \square \)

Acknowledgement. The author would like to thank referees for their helpful comments and suggestions.

References


Department of Mathematics
Hoseo University, Asan
Choongnam 337-850, Korea