1. Introduction

We shall, for the most part, use the terminology of [2]. Graphs will be finite or infinite, but have no loops or multiple edges. For a vertex $v$ of $G$, denote by $N_G(v)$ the set of vertices adjacent to $v$ in $G$, and by $d_G(v)$ the cardinal number of $N_G(v)$. An $x,y$-path is a path joining vertices $x$ and $y$ in $G$, and in this case $x$ and $y$ are called the endvertices of the path. A path $P$ is one-side infinite if it contains infinitely many vertices and $d_P(x) = 1$, for only one vertex $x$ in $P$. In this case the vertex $x$ is said to be the endvertex of $P$.

Let $G$ be a plane graph and let $C$ be a cycle in $G$. We denote by $\overline{C}$ the subgraph of $G$ consisted of the vertices and the edges lying on $C$ and lying in the interior of $C$. A plane graph $H$ is a circuit graph, following D.Barnette, if there exists a cycle $C$ in a 3-connected plane graph such that $H = \overline{C}$. A circuit graph $H$ is triangulated if all facial cycles of $H$, up to the outer cycle, are triangles.

A triangulation $G$ is a countable locally finite plane graph, of which edges are contained in two non-separating triangles. If a representation of the graph $G$ contains no vertex- or edge-accumulation points, then $G$ is called a strong triangulation.

Whitney [8] proved every finite 4-connected maximal planar graph has a Hamiltonian cycle, and Tutte [7] and Thomassen [6] extended his result to all 4-connected planar graphs. In particular, Thomassen [6] showed that every 4-connected planar graph is Hamiltonian-connected, i.e., it has a Hamiltonian path connecting any two prescribed vertices. On the other hand, Dillencourt [3] observed the condition for internally maximal planar graphs to have a Hamiltonian cycle, and so he proved that every triangulated circuit graph without separating triangles, which contains at most three chordal edges, is Hamiltonian.
A simplified proof of Whitney's theorem and a linear algorithm for finding a Hamiltonian cycle in such a graph, can also be found in [1].

Nash-Williams ([4], see also in [5]) conjectured that this theorem is also true for all infinite 4-connected planar graphs, i.e., every infinite 4-connected planar graph has a one-side infinite Hamiltonian path.

In this paper Whitney's theorem will be extended to the infinite strong triangulations under the corresponding hypothesis, which is a part of Nash-williams' conjecture.

Namely, we prove the following theorem.

**Theorem.** Let $G$ be a 4-connected infinite strong triangulation. Then there exists a one-side infinite Hamiltonian path in $G$ originating from any prescribed vertex.

For the proof, important tools are the structure theorem (in section 2), Whitney’s theorem (in section 3) and the so-called König’s Unendlichkeitslemma, which is stated below:

**Lemma (König).** Let $\{P_1, P_2, P_3, \ldots\}$ be an infinite sequence of disjoint non-empty finite sets and $R$ be a relation in $\mathcal{P} := \bigcup_{j=0}^{\infty} \mathcal{P}_j$, such that

$$\forall j \in \mathbb{N}, \forall P' \in \mathcal{P}_{j+1}, \exists P \in \mathcal{P}_j \text{ such that } (P, P') \in R.$$  

Then there exists an infinite sequence of paths $\{P_1, P_2, P_3, \ldots\}$ such that $P_j \in \mathcal{P}_j$ and $(P_j, P_{j+1}) \in R$.

To investigate the structure of an infinite strong triangulation, we in addition have to define several important conditions.

Let $C$ and $C'$ be two disjoint cycles in an infinite strong triangulation $G$, where $C$ lies in the interior of $C'$. A $(C, C')$-ring is a subgraph of $G$, which consists of not only $C$ and $C'$ but also the vertices and edges lying between $C$ and $C'$. For a $(C, C')$-ring $R$, a bridge of $R$ is either an edge of $R$ joining $C$ and $C'$ (such a bridge is called a chordal edge, following Dillencourt [3]), or it is a connected component of $R - (C \cup C')$ together with all edges of $R$ joining this component to $C \cup C'$. A $(C, C')$-ring $R$ is *normal* if it satisfies the following properties:

1. $C$ and $C'$ are induced cycles.
2. $|V(B) \cap V(C')| \leq 2$, for any bridge $B$ of $R$. 

2. Structure of infinite strong triangulations

**Lemma 1.** Let $C$ be an induced cycle of an infinite strong triangulation $G$. Then there exists a cycle $C'$ such that the $(C, C')$-ring is normal.

**Proof.** First, we construct a cycle $C'$ in $G$ satisfying the hypothesis of this lemma.

Let $F := \{ J | J \text{ is a facial cycle in } G \text{ such that } V(J) \cap V(C) \neq \emptyset \}$ and let $E$ be the set of all vertices of the cycles in $F$. Then we can see that $|E| < \infty$, since $E$ contains only finite cycles and $F$ is also finite. Furthermore, set $H := G[E]$, i.e. $H$ is the induced subgraph of $G$ containing all elements of $E$, and let $C'$ be its outer cycle of $H$. We will now show the $(C, C')$-ring $R$ is normal.

As an induced subgraph $H$ of $G$, $C'$ is an induced cycle. The assertion (3) is also obvious from the assumption. To show that $C$ and $C'$ are disjoint, we assume: $\exists x \in V(C) \cap V(C')$. Let $y$ be a vertex on $C'$ adjacent to $x$. Then, from the fact that all facial cycles in $G$ are triangles, we can find a facial cycle $J = \{ x, y, z \}$ such that $yz \not\in E(C')$. But since the cycle must be contained in $F$ (since $V(J) \cap V(C) \neq \emptyset$), it follows that $y, z \in E$. Hence we have $yz \in E(H)$, which contradicts our construction of $C'$.

It remains to be shown that $|V(B) \cap V(C')| \leq 2$ for every bridge $B$ of $R$. Suppose there exists a bridge $B$ such that $V(B) \cap V(C') = \{ y_1, \ldots, y_r \}$, $r \geq 3$. Since $B$ is not a chordal edge and $V(B) \setminus V(C \cup C') \neq \emptyset$, it follows that there exists a $y_1, y_r$-path $P$ in $B - (C \cup \{ y_2, \ldots, y_{r-1} \})$. Thus the facial cycle in $R$ containing the edge $y_ky_{k+1}$ ($k = 1, \ldots, r - 1$) is not contained in $F$, and therefore it holds that $y_k \not\in E$, $k = 2, \ldots, r - 1$, which also contradicts our construction of $C'$. $\Box$

**Remark.** We can prove that such a cycle $C'$ is unique for a given induced cycle $C$.

**Lemma 2.** For any cycle $C$ of an infinite strong triangulation, the induced subgraph $\overline{C}$ is a triangulated circuit graph.
Proof. As every strong triangulation has a vertex-accumulation point free representation, $\overline{C}$ is a finite subgraph, and hence it is a circuit graph. It is also obvious that $\overline{C}$ is triangulated. □

**PROPOSITION 3.** Let $G$ be a 4-connected strong triangulation. Let $x_0$ be a vertex of $G$ and let $C_0$ be the cycle of $G$ consisting of the vertices adjacent to $x_0$. Then there exists a sequence of induced cycles \{$C_0,C_1,C_2,\ldots$\} which holds the following properties:

1. The $(C_{j-1},C_j)$-ring is normal for all $j \in \mathbb{N}$.
2. $V(G) = V(\bigcup_{j=0}^{\infty} C_j)$.

**Proof.** It is clear that $C_0$ is an induced cycle by the fact of 4-connectedness of $G$. For $j \in \mathbb{N}$ the existence of $C_j$, related to $C_{j-1}$, satisfying the condition (1) follows from lemma 1. It remains only to show that the resulting cycles \{$C_0,C_1,C_2,\ldots$\} hold the condition (2).

Let $x \in V(G)$ be an arbitrary vertex. Since $C_{j-1}$ lies in the interior of $C_j$ ($j \in \mathbb{N}$), it follows that $x \in V(\overline{C}_{n_x})$, where $n_x$ is a metric distance between $x$ and $x_0$. Because of $V(\overline{C}_{n_x}) \subseteq V(\bigcup_{j=0}^{\infty} \overline{C}_j)$, we have $V(G) \subseteq V(\bigcup_{j=0}^{\infty} \overline{C}_j)$. Since it holds clearly that $V(G) \supseteq V(\bigcup_{j=0}^{\infty} \overline{C}_j)$, we can conclude $V(G) = V(\bigcup_{j=0}^{\infty} \overline{C}_j)$. □

**REMARK.** We can also prove that, for an arbitrary given vertex $x_0$, such a sequence of induced cycles with the condition (1)-(2) is unique.

Let $C$ be an induced cycle in an infinite strong triangulation $G$. According to lemma 2 we can construct a cycle $C'$ in $G$ such that $(C,C')$-ring $R$ is normal. We let $F$ be the set of all chordal edges of $R$ and let $BG(R) := (C \cup C') \cup F$. Then we have exactly $|F|$ facial cycles in $BG(R)$, up to the interior of $C$ and the exterior of $C'$. For a facial cycle $J$ of $BG(R)$ the induced subgraph $\overline{J}$ of $G$ is called a chamber of $R$. If $J = \overline{J}$, then the chamber $J$ is empty. Clearly in the interior of a chamber lies at most one bridge of $R$ since $G$ is maximal planar.

Now we let $G$ be 4-connected and let $L$ be a nonempty chamber of $R$. Because of the conditions (2) and (3) in definition of normality, $L$ must be one of following two types:

(i) $|V(L) \cap V(C')| = 1$,

(ii) $|V(L) \cap V(C')| = 2$.

In the former case we say that $L$ is of type 1 and in the latter case that $L$ is of type 2.
Hamiltonian Paths in Infinite Strong Triangulations

3. Whitney’s theorem and its extensions

The following notations are useful for the concept and proof of Whitney’s lemma and its corollaries.

A path \( P \) on \( C \) is \textit{wh-induced} if there exists no edge \( xy \in E(H) \setminus E(C) \), \( x, y \in V(P) \). For distinct vertices \( u, v \) (resp. \( u, v, w \)) on \( C \), we say that \( (H, u, v) \) (resp. \( (H, u, v, w) \)) satisfies condition \( W1 \) (resp. \( W2 \)) if the two \( u, v \)-paths (resp. the \( u, v, w \) and \( w, u \)-path) on \( C \) are wh-induced. Note that according to our definition \( (H, u, v) \) satisfies \( W1 \) if and only if \( (H, u, v, w) \) satisfies \( W2 \) for every vertex \( w \) on \( C \).

\[ \text{LEMMA 4 (H.WHITNEY).} \]
Let \( H \) be a triangulated circuit graph without separating triangles and let \( C \) be its outer cycle. Finally let \( u \) and \( v \) be two distinct vertices on \( C \). If \( (H, u, v) \) satisfies the condition \( W1 \) or if \( (H, u, v, w) \) satisfies \( W2 \) for some vertex \( w \) on \( C \), then \( H \) contains a Hamiltonian \( u, v \)-path.

\[ \text{Proof.} \] See in [8]. \[ \square \]

\[ \text{LEMMA 5.} \]
Let \( H \) be a 3-connected triangulated circuit graph without separating triangles and let \( C \) be its outer cycle with \( |V(C)| \geq 4 \).

(1) Let \( y \in V(C) \), and let \( u, v \) be the vertices adjacent to \( y \) on \( C \). Then there exists a Hamiltonian \( u, v \)-path in \( H - y \).

(2) Let \( yy' \in E(C) \) and let \( u \) (resp. \( v \)) be the vertex adjacent to \( y \) (resp. \( y' \)) on \( C \) such that \( u \neq y' \) and \( v \neq y \). Then there exists a Hamiltonian \( u, v \)-path in \( H - \{y, y'\} \).

\[ \text{Proof.} \] (1) Set \( H' := H - y \). Then \( H' \) clearly is a triangulated circuit graph, because it is 2-connected. Let \( J \) be the outer cycle of \( H' \) and let \( J_1 \) and \( J_2 \) be the \( u, v \)-paths on \( J \) with \( J_1 = C - y \). Then the vertices of \( J_2 \) are identical to the vertices adjacent to \( y \) in \( H \) since \( H \) is triangulated. Note that \( J_1 \) is wh-induced. We will show \( J_2 \) also is wh-induced.

Suppose that there is an edge \( xx' \) in \( E(H) \setminus E(J_2) \) contained in the interior of \( J \). Then the vertices \( \{x, x', y\} \) separate \( H \) in two components, and hence \( H \) contains a separating triangle since \( xx', x'y \in E(H) \). So we have a contradiction to the hypothesis of this lemma.

Therefore \( (H', u, v) \) satisfies \( W1 \), and so \( H' = H - y \) contains a Hamiltonian \( u, v \)-path, by the Whitney’s lemma.
(2) From $yy' \in V(C)$ and $|V(C)| \geq 4$, $H' := H - \{y, y'\}$ is 2-connected, and from this it is a triangulated circuit graph. Let $J$ be the outer cycle of $H'$, and let $J_1$ and $J_2$ be the $u, v$-paths on $J$ such that $J_1 = C - \{y, y'\}$. Then we have $V(J_2) = N_G(\{y, y'\})$. We first note that $J_1$ is $wh$-induced. Let us consider the path $J_2$.

Since $H'$ is triangulated we can easily verify that there exists a $u, v$-path $J'$ such that $J' \subseteq V(J_1)$.

(i) $V(J'_2) \subseteq V(J_2)$,
(ii) $J'_2$ is induced path if $|V(C)| \geq 5$,
and $J'_2 \cup \{uv\}$ is induced cycle if $|V(C)| = 4$.

If $V(J'_2) = V(J_2)$, then $J_2$ is $wh$-induced, and so $(H', u, v)$ satisfies $W_1$.

Now assume that $V(J'_2) \subset V(J_2)$. From the fact $H$ contains no separating triangles, it is easy to see that there exists only one edge $e \in E(H)$ such that $e \in E(J'_2) \setminus E(J_2)$. Let $w$ be the vertex of $J_2$ such that $\{y, y', w\}$ constitutes a facial cycle of $H$. Then, as in the proof of (1), it can be verified that the $u, w$- and $v, w$-path on $J_2$ are wh-induced. Therefore $(H', u, v, w)$ satisfies $W_2$, and hence, in both cases, we can find a Hamiltonian $u, v$-path in $H' = H - \{y, y'\}$ by Whitney's lemma.

**Lemma 6.** Let $H$ be a triangulated circuit graph without separating triangles and let $C$ be its outer cycle. Let $u, v \in V(C), u \neq v$, and $e \in E(C)$ arbitrary (but $e \neq uv$ if $uv \in E(C)$). If $(H, u, v)$ satisfies the condition $W_1$, then $H$ has a Hamiltonian $u, v$-path which contains the edge $e$.

**Proof.** Let $e := xy \in E(C)$ and let $w$ be a further vertex not in $H$. We construct a graph $\tilde{H}$ as follows:

$$V(\tilde{H}) := V(H) \cup \{w\} \quad E(\tilde{H}) := E(H) \cup \{xw, yw\}.$$ 

Then $\tilde{H}$ again is triangulated and $(H, u, v, w)$ further satisfies $W_2$, and hence there exists a Hamiltonian $u, v$-path $\tilde{P}$ in $\tilde{H}$ by Whitney's lemma. Let $V(P) := V(\tilde{P}) \setminus \{w\}$ and $E(P) := E(\tilde{P}) \cup \{xy\} \setminus \{xw, yw\}$. Since $\tilde{P}$ must contain the edge $xw$ and $yw$, the $u, v$-path $P$ is Hamiltonian in $H$ containing the edge $e = xy$. □
4. Proof of the main theorem

Let $R$ be a normal $(C,C')$-ring in a 4-connected infinite strong triangulation $G$. We choose an arbitrary vertex $y_0$ in $V(C')$. Let $\bar{y}$ be the first vertex adjacent to $y_0$ on $C'$, counterclockwise, and set $M := N_G(\bar{y}) \cap V(C)$. We note that $M$ is non-empty since $G$ is maximal planar. Let $x_1$ be the first vertex in $M$, also counterclockwise, and set $\{x_1, \ldots, x_k\} \subseteq V(C) \cap N_G(C')$. (i.e. for every $i \in \{1, \ldots, m\}$, there exists a vertex $y \in V(C')$ with $x_i y \in E(G)$, and conversely). Then for every $i \in \{1, \ldots, m\}$ and for each pair $\{x_i, x_{i+1}\}$ we can find exactly one chamber $L_i$ such that $x_i, x_{i+1} \in V(L_i)$. Let $x_0$ be the vertex adjacent to $x_m$ on $C$ lying between $x_m$ and $x_0$. (If $x_m x_1 \in E(C)$ we let $x_0 = x_1$). We will prove there exists a Hamiltonian $x_0, y_0$-path in $R - (V(C') \setminus \{y_0\})$.

(1) The chamber $L_i (i = 1, \ldots, m - 1)$.

Case 1: $L_i$ is of type 1:

Let $y \in V(L_i) \cap V(C')$. If $L_i$ is empty we let $P_i := \{x_i, x_{i+1}\}$. Otherwise $L_i$ clearly is a 3-connected triangulated circuit graph without separating triangles. Since $x_i$ and $x_{i+1}$ are adjacent to $y$ on the outer cycle of $L_i$ we can find a Hamiltonian $x_i, x_{i+1}$-path $P_i$ in $L_i - y$ by lemma 5 (1).

Case 2: $L_i$ is of type 2:

Let $y, y' \in V(L_i) \cap V(C')$. Since $R$ is normal, $yy'$ must be an edge of $L_i$. Analogously it can be verified that $L_i$ satisfies the hypothesis of (2) in lemma 5. Therefore we can also find a Hamiltonian $x_i, x_{i+1}$-path $P_i$ in $L_i - \{y, y'\}$.

(2) The chamber $L_m$.

Let $J$ be the outer cycle of $L_m$. Because of the choice of $x_1$, it is clear that $y_0 x_m, x_1 y \in E(G)$. We will construct a Hamiltonian $x_1, y_0$-path $\bar{P}$ in $L_m$ (resp. $L_m - \bar{y}$) containing the edge $x_m x_0$ if $L_m$ is of type 1 (resp. type 2), where the vertex $x_0$ is defined at the beginning of this section.

Case 1: $L_m$ is of type 1:

Let $V(L_m) \cap V(C') := \{y_0\}$. If $L_m$ is empty, then we let $\bar{P} = L_m - \{x_1 y_0\}$. Otherwise $L_m$ is 3-connected and $|V(J)| \geq 4$. Because $x_1 y_0 \neq x_m x_0$, $L_m$ satisfies the hypothesis of lemma 5 (corresponding
to the vertices $x_1, y_0$ and the edge $x_m x_0$). Therefore there exists a Hamiltonian $x_1, y_0$-path $\bar{P}$ in $L_m$ containing the edge $x_m x_0$.

Case 2: $L_m$ is of type 2:

Let $V(L_m) \cap V(C') =: \{y_0, \bar{y}\}$. In this case $L_m$ is not empty, so it is 3-connected. As in the proof of lemma 5, $(L_m - \bar{y}, x_1, y_0)$ satisfies the condition $W1$, so, by Whitney’s lemma, there exists a Hamiltonian $x_1, y_0$-path $\bar{P}$ in $L_m - \bar{y}$ containing the edge $x_m x_0$.

In each case we let $P_0$ be the $x_0, x_1$-path of $\bar{P}$ and $P_m$ be the $x_1, x_m$-path of $\bar{P}$. Then $V(P_0) \cup V(P_m) = V(\bar{P})$ and $E(P_0) \cup E(P_m) = E(\bar{P}) \setminus \{x_m x_0\}$ since $\bar{P}$ contains the edge $x_m x_0$.

Now we sumerize all chambers $L_1, \ldots, L_{m-1}, L_m$. For a given normal $(C, C')$-ring $R$ in $G$ and for an arbitrary given vertex $y_0$ on $C'$, the chambers $L_1, \ldots, L_m$ are fixed. From (1) and (2) we have $m + 1$ paths $P_0, P_1, \ldots, P_m$ in $R$, such that:

1. for $i = 0, \ldots, m - 1$ the endvertices of $P_i$ are $x_i$ and $x_{i+1}$, and those of $P_m$ are $x_m$ and $y_0$.
2. $V(P_i) = V(L_i - C')$ for $i = 0, \ldots, m - 1$, and $V(P_0) \cup V(P_m) = V(L_m)$.

Let $P := \bigcup_{i=0}^{m} P_i$. Then $P$ is a $x_0, y_0$-path in $R$ which covers all vertices of $(R - C') \cup \{y_0\}$. Thus we have:

**Proposition 7.** Let $R$ be a normal $(C, C')$-ring in a 4-connected infinite strong triangulation and let $x_0, y_0$ be the vertices defined at the beginning of this section. Then there exists a $x_0, y_0$-path in $R$ which covers all vertices of $(R - C') \cup \{y_0\}$. □

We can now prove the main theorem of this paper with the aid of König’s lemma.

Let $G$ be a 4-connected infinite strong triangulation and $x_0$ an arbitrary given vertex of $G$. We let $C_0$ be the induced cycle of $G$ consisting of the vertices adjacent to $x_0$. Then, by proposition 3, we have a sequence of induced cycles $\{C_0, C_1, C_2, \ldots\}$ satisfying the same conditions (1)–(2) in the proposition.

For $j \in \mathbb{N}$, let $R_j$ be the $(C_{j-1}, C_j)$-ring and let $P_j$ be the set of all
paths in $R_j$ such that:

$$ P \in \mathcal{P}_j \quad \text{if and only if} \quad \begin{cases} 
i P \text{ is a } x, y\text{-path in } R_j \\ \text{with } x \in V(C_{j-1}) \text{ and } y \in V(C_j), \\
i E(P) = E(R_j - C_j) \cup \{y\}. \end{cases} $$

We will further define a relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$, where $\mathcal{P} := \bigcup_{j=1}^{\infty} \mathcal{P}_j$:

$$(P, P') \in \mathcal{R} \quad \text{if and only if} \quad \begin{cases} 
i \exists j \in \mathbb{N}; \ P \in \mathcal{P}_j \text{ and } P' \in \mathcal{P}_{j+1}, \\
i P \text{ and } P' \text{ have a common endvertex}. \end{cases}$$

We will show that the relation $\mathcal{R}$ holds the hypothesis of König’s lemma. Clearly we have $\mathcal{P}_j \neq \emptyset$ and $|\mathcal{P}_j| < \infty$ for all $j \in \mathbb{N}$. For any $j \in \mathbb{N}$, let $P' \in \mathcal{P}_{j+1}$ be an arbitrary element. Then, by the definition of $P'$, one of its endvertices of $P'$, say $x'$, is contained in $C_j$ and the another in $C_{j+1}$. By proposition 7, we can find a $x, x'$-path $P$ in $R_j$ with $x \in V(C_{j-1})$ and $V(P) = V(R_j - C_j) \cup \{x'\}$, and from this we have $P \in \mathcal{P}_j$ and $(P, P') \in \mathcal{R}$. Thus, by König’s lemma, there exists an infinite sequence of paths $\{P_1, P_2, \ldots\}$ such that $P_j \in \mathcal{P}_j$ and $(P_j, P_{j+1}) \in \mathcal{R}$ for all $j \in \mathbb{N}$. We now let $x_1$ be the endvertex of $P_1$ lying on $C_0$ and set $P_0 := \{x_0x_1\}$. Then $P := \bigcup_{j=0}^{\infty} P_j$ clearly is a one-side infinite path in $G$. Because of $V(G) = \{x_0\} \cup V(\bigcup_{j=1}^{\infty} P_j) = V(\bigcup_{j=0}^{\infty} C_j)$ by proposition 3, $P$ is a Hamiltonian path in $G$ originating from the endvertex $x_0$, and this completes the proof of the main theorem.

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Department of Mathematics
Han-Shin University
Osan-shi 447-791, Korea