ON THE MINIMIZERS OF CERTAIN SINGULAR CONVEX FUNCTIONALS

HI JUN CHOE

1. Introduction

In this paper the minimizers of singular functionals are considered. Suppose that $O$ is a bounded, open, convex subset of $\mathbb{R}^n$ and $f : O \to \mathbb{R}$ is smooth and uniformly strictly convex. Suppose further that $f \geq 0$ and

$$\lim_{P \to \partial O} f(P) = \infty.$$ 

Set $f(P) = \infty$ for all $P \in \mathbb{R}^n \setminus O$.

For example, $f$ can be one of the following:

$$f(P) = \frac{1}{1 - |P|^2}, \quad O = \{P : |P| < 1, \ P \in \mathbb{R}^n\}$$

or

$$f(P) = \frac{1}{1 - P_1^2} + \frac{1}{1 - P_2^2}, \quad O = (-1, 1) \times (-1, 1).$$

Consider the functional

$$I(u) = \int_{\Omega} f(Du) dx$$

defined for appropriate $u : \Omega \to \mathbb{R}$, where $\Omega$ is a bounded, open subset of $\mathbb{R}^n$ with smooth boundary.

Suppose that $u_0 \in W^{1,2}(\Omega)$ and $I(u_0) < \infty$. Then $u_0$ is Lipschitz on the closure of $\Omega$ and it is relatively easy to see that there exists a unique $u \in u_0 + W_0^{1,2}(\Omega)$ such that

$$I(u) \leq I(v)$$

Received May 9, 1992.

The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1991. Project No. BSRI-91-115.
for all \( v \in u_0 + W^{1,2}_0(\Omega) \). Of course, \( u \) is Lipschitz on the closure of \( \Omega \) with \( u = u_0 \) on \( \partial \Omega \) and minimizes \( I \) among all such functions. The first question addressed is that of the regularity of \( Du \).

Suppose that \( u_0 \) satisfies the following "bounded slope condition": there exists a constant \( M \) such that for each point \( x_0 \in \partial \Omega \), there exist linear functions \( \pi^\pm \) such that

\[
\pi^-(x - x_0) \leq u_0(x) - u_0(x_0) \leq \pi^+(x - x_0)
\]

for all \( x \in \partial \Omega \). Then it is shown that \( u \in C^{1,\alpha}(\Omega) \) for any \( \alpha \in (0,1) \). In fact, \( f(Du) \) is bounded, and since \( f \) is smooth, it follows that \( u \in C^\infty(\Omega) \).

This seems to be the first regularity result of this type, i.e., where the function \( f \) exhibits this type singular behavior. The study of this question is motivated by models for hyperelastic materials (see Ball [1]) in which one is lead to consider minimizers of functionals over vector-valued mappings where the integrand exhibits a certain type of singular behavior.

In case \( u_0 \) does not satisfy the bounded slope condition given above, then the corresponding minimizer need not be in \( C^1 \). An example is given, in case \( n = 2 \),

\[
f(P_1, P_2) = (1 - P_1^2)^{-\alpha} + (1 - P_2^2)^{-\alpha}, \quad (0 < \alpha < 1)
\]

with \( \Omega \) an open ball in \( \mathbb{R}^2 \). The minimizer fails to be in \( C^1 \) exactly on a line joining two points of \( \partial \Omega \).

For systems, it is known that minimizers need not be \( C^1 \) even if \( f \) is uniformly strictly convex on all of \( \mathbb{R}^n \) with bounded second derivatives. In case \( n = 1 \) and \( f \) depends on \( u \) and \( Du \), J. M. Ball and V. Mizel[2] have given examples showing that singularities can occur in the interior of \( \Omega \). As far as we know the examples given here are the first in the higher-dimensional scalar case showing that singularities can occur in the interior even if \( f \) depends only on \( Du \).
ACKNOWLEDGEMENT. The problem was asked by Professor L. C. Evans and R. Gariepy. Especially the author would like to thank Professor R. Gariepy. The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1991 Project No. BSRI-91-115

2. Existence and uniqueness

Suppose that $O$ is a bounded open convex subset of $\mathbb{R}^n$ and that $f : O \to \mathbb{R}$ is $C^2(O)$ and for some constant $\lambda > 0$,

$$
\frac{\partial^2}{\partial P_i \partial P_j} f(P) \xi_i \xi_j \geq \lambda |\xi|^2
$$

for all $P \in O$, $\xi \in \mathbb{R}^n$. Suppose further that

$$
\lim_{P \to \partial O} f(P) = \infty
$$

and $f(P) = \infty$ for all $P \in \mathbb{R}^n \setminus O$.

Suppose $\Omega$ is an open connected subset of $\mathbb{R}^n$. Suppose $u_0 \in W^{1,1}(\Omega; \mathbb{R})$ and

$$
I[u_0] = \int_\Omega f(Du_0) dx < \infty.
$$

Lemma 1. Let $K = \{ v \in u_0 + W^{1,1}(\Omega) : I[v] < \infty \}$. Then $K$ is convex.

Proof. Let $v_1, v_2 \in K$, then for each $0 \leq t \leq 1$, $tv_1 + (1-t)v_2 \in u_0 + W^{1,1}_0$ and $I[v_1], I[v_2] < \infty$. Since $f$ is convex, we have

$$
I[tv_1 + (1-t)v_2] = \int_\Omega f(tDv_1 + (1-t)Dv_2) dx
$$

$$
\leq t \int_\Omega f(Dv_1) dx + (1-t) \int_\Omega f(Dv_2) dx < \infty.
$$

So $tv_1 + (1-t)v_2 \in K$ and $K$ is convex.

Since $O$ is bounded, $K$ is a bounded subset of $W^{1,p}$ for all $1 \leq p \leq \infty$. Since any convex function is bounded below, we assume that $f$ is nonnegative and $f(0) = 0$ is the minimum of $f$ in $O$. The next theorem proves that $I$ is weakly sequentially lower semicontinuous in $W^{1,1}(\Omega)$. 
THEOREM 1. Suppose that \(u, u_n \in K\) for each \(n\) and \(u_n \rightharpoonup u\) weakly in \(W^{1,1}(D)\) for each \(D \subset \Omega\). Then

\[
I[u] \leq \liminf_{n \to \infty} I[u_n].
\]

**proof.** Let \(d = \text{dist}(D, \partial \Omega)\) and \(\phi\) be a nonnegative smooth function supported in the unit ball such that

\[
\int_{\mathbb{R}^n} \phi(x) \, dx = 1.
\]

Define \(w_\rho(x)\) by

\[
w_\rho(x) = \frac{1}{\rho^n} \int_{\mathbb{R}^n} \phi(\frac{x-y}{\rho}) w(y) \, dy
\]

for each function \(w \in W^{1,1}(\Omega)\) and \(\rho > 0\). Since \(Du_\rho \rightharpoonup Du\) almost everywhere in \(D\) as \(\rho \to 0\) and \(f\) is continuous, we have

\[
f(Du_\rho) \to f(Du)
\]

almost everywhere in \(D\) as \(\rho \to 0\). Since \(f\) is nonnegative,

\[
I[u : D] = \int_D f(Du) \, dx \leq \liminf_{\rho \to 0} \int_D f(Du_\rho) \, dx
\]

by Fatou's lemma. From Jensen's inequality, we have

\[
f(Du_\rho) \leq f(Du)_\rho
\]

and

\[
f(Du_{n, \rho}) \leq f(Du_n)_\rho
\]
on \(D\), for each \(n\) and \(\rho < d\).

So we see that

\[
I[u_{n, \rho} : D] \leq \int_D f(Du_{n, \rho}) \, dx \leq \int_\Omega f(Du_n) \, dx.
\]
Since $Du_n \rightharpoonup Du$ weakly in $L^1(D')$ for each $D \subset D' \subset \subset \Omega$, $Du_{n,\rho} \rightharpoonup Du_\rho$ pointwisely in $D \subset \subset \Omega$ as $n \to \infty$. Thus, combining the previous inequalities, we have

$$I[u_\rho : D] = \int_D f(Du_\rho)dx$$

$$= \int_D \lim_{n \to \infty} \inf f(Du_{n,\rho})dx$$

$$\leq \lim_{n \to \infty} \inf \int_D f(Du_{n,\rho})dx$$

$$\leq \lim_{n \to \infty} \inf \int_\Omega f(Du_n)dx.$$ 

So we have

$$I[u : D] = \int_D f(Du)dx$$

$$\leq \lim_{\rho \to 0} \inf \int_D f(Du_\rho)dx$$

$$\leq \lim_{n \to \infty} \inf \int_\Omega f(Du_n)dx.$$ 

Since $D$ is chosen arbitrarily, we have

$$I[u : \Omega] = I[u] \leq \lim_{n \to \infty} \inf I[u_n]$$

and $I$ is lower semicontinuous.

Now we prove the existence and uniqueness of the minimizer. The theorem follows essentially from the weak compactness of the bounded subset of $W^{1,2}(\Omega)$.

**THEOREM 2.** Let $\mu = \inf_{v \in K} I[v]$. Then there is a unique $u \in K$ such that

$$I[u] = \mu.$$ 

**proof.** We note that $\mu$ is a finite number, since $I$ is convex and $u_0 \in K$. Let $\{u_n\}$ be a sequence in $K$ such that $I[u_n] \to \mu$ as $n \to \infty$. Since $K$ is a bounded subset of $W^{1,2}(\Omega)$, there is a subsequence $\{u_{n_k}\}$
such that $u_{n_k} \rightharpoonup u$ weakly in $W^{1,2} (\Omega)$ for some $u \in W^{1,2} (\Omega)$ as $k \to \infty$. We see that $u - u_0 \in W^{1,2}_0 (\Omega)$. From the lower semicontinuity of $I$ we have

$$I[u] \leq \lim_{k \to \infty} \inf I[u_{n_k}] = \mu$$

So $I[u] = \mu$ from the fact that $\mu = \inf_{v \in K} I[v]$.

We prove the uniqueness by using the strict convexity of $f$ and a variational inequality which the minimizer $u$ satisfies. First we show that $u$ satisfies a variational inequality. Suppose that $v \in K$. Then $I[v] < \infty$. Since $f$ is convex,

$$f(Dv(x)) - f(Du(x)) \geq \frac{f(Du(x) + t(Dv(x) - Du(x))) - f(Du(x))}{t}$$

for all $x \in \Omega$ and $0 < t \leq 1$. Moreover we see that

$$F_t(x) = \frac{f(Du(x) + t(Dv(x) - Du(x))) - f(Du(x))}{t}$$

is monotone decreasing as $t \to 0$ and $F_t(x)$ converges to $f_{\pi_t}(Du(x))$ $(Dv - Du)$ for almost all $x \in \Omega$. Since $F_t(x) \leq f(Dv(x)) - f(Du(x))$ for all $x \in \Omega$ and $F_t$ converges to $f_{\pi_t}(Du(x))(Dv - Du)$ monotonically as $t \to 0$, by the monotone convergence theorem, we have

$$\lim_{t \to 0} \int_{\Omega} F_t(x) dx = \int_{\Omega} f_{\pi_t}(Du)(Dv - Du) dx.$$

Since $I[v] - I[u] \geq \int_{\Omega} F_t(x) \geq 0$ for all $0 < t \leq 1$,

$$I[v] - I[u] \geq \int_{\Omega} f_{\pi_t}(Du(x))(Dv - Du) dx \geq 0$$

for all $v \in K$.

Let $I[u] = I[v]$ for some $v \in K$. Then since $f$ is strictly convex, we
have

\[ 0 = I[u] - I[v] \]

\[ = \int_\Omega [f(Du) - f(Dv)] \, dx \]

\[ = \int_\Omega f_{P_1}(Du)(D_iv - D_iu) \, dx \]

\[ + \int_\Omega \int_0^1 (1 - t)f_{P_1,P_2}(Du + t(Dv - Du))dt \]

\[ \times (D_iv - D_iu)(D_jv - D_ju) \, dx \]

\[ \geq \frac{1}{2} \lambda \int_\Omega |Du - Dv|^2 \, dx. \]

Since \( u - v \in W^{1,2}_0(\Omega) \), from Sobolev's inequality,

\[ \| u - v \|_2 \leq C \| Du - Dv \|_2 = 0 \]

for some \( C \), where \( \| u \|_2 \) is \( L^2(\Omega) \) norm of \( u \).

3. Approximation

We approximate \( f \) with functions \( f^\rho \) which grow quadratically by using the implicit function theorem.

Let \( E^\rho = \{ P \in \mathbb{R}^n : f(P) \leq \rho \} \). Then \( E^\rho \) is a strictly convex, bounded and closed subset of \( \mathbb{R}^n \).

Now we construct a uniformly strictly convex function with quadratic growth. First we recall the Implicit Function Theorem.

**Lemma 2.** Let \( g \in C^2 \) and \( D_y g(x_0, y_0) \neq 0 \). Then there exists a function \( h(y) \) such that \( x_0 = h(y_0) \) and \( g(h(y), y) = 0 \) in some neighborhood \( U \) of \( y_0 \). Moreover \( h \in C^2(U) \).

The following theorem is fundamental to the approximation.

**Theorem 3.** Suppose \( g \in C^2(\mathbb{R}^n) \) and

\[ g_{P_i P_j}(P)\xi_i \xi_j \geq \nu |\xi|^2 \]
for some $\nu > 0$ and all $P, \xi \in \mathbb{R}^n$. Suppose further that $g(0) = 0$ is the minimum. Define by $r(P) > 0$

$$g\left(\frac{P}{\sqrt{r(P)}}\right) = c > 0$$

for all $P \in \mathbb{R}^n \setminus \{0\}$. Then $r(P) \in C^2(\mathbb{R}^n \setminus \{0\})$ and

$$\nu_1 \left| \xi \right|^2 \leq r \phi_1(P) \phi_j \xi_j \leq \nu_2 \left| \xi \right|^2$$

for all $P, \xi \in \mathbb{R}^n \setminus \{0\}$. Moreover $\nu_1$ and $\nu_2$ depend only on $c$ and $\nu$.

**Proof.** We see that $g$ is radially strictly increasing. Since we are assuming $r > 0$, $r(P)$ is well defined for all $P \in \mathbb{R}^n \setminus \{0\}$. By differentiating $g\left(\frac{P}{\sqrt{r}}\right)$ with respect to $r$, we have

$$\frac{\partial}{\partial r} g\left(\frac{P}{\sqrt{r}}\right) = g_1\left(\frac{P}{\sqrt{r}}\right) P_i \left(\frac{1}{2 \sqrt{r^3}}\right) \neq 0$$

for all $P \in \mathbb{R}^n \setminus \{0\}$. So, from Lemma 2, $r(P) \in C^2$ and

$$g\left(\frac{P}{\sqrt{r(P)}}\right) = c.$$

Define

$$g_i = g_P\left(\frac{P}{\sqrt{r}}\right), \quad g_{ik} = g_{Pi} P_k\left(\frac{P}{\sqrt{r}}\right), \quad r_i = r_P\left(\frac{P}{\sqrt{r}}\right), \quad r_{ik} = r_{Pi} P_k\left(\frac{P}{\sqrt{r}}\right).$$

We see that

$$\frac{\partial}{\partial P_i} g\left(\frac{P}{\sqrt{r}}\right) = g_i \frac{1}{\sqrt{r}} - \frac{1}{2} g_{j} P_j \frac{r_i}{r^3} = 0,$$

and hence that

$$r_i = 2 \frac{g_i r}{g_j P_j}.$$
Setting $T = g_j P_j$ and differentiating with respect to $P_k$,

$$
\frac{1}{2} r_{ik} T^2 = T g_{ik} \sqrt{r} - \frac{1}{2} T g_{il} P_l \frac{r_k}{\sqrt{r}} + T g_i r_k \\
- g_i g_{jk} P_j \sqrt{r} + \frac{1}{2} g_{ji} P_l g_i P_j \frac{r_k}{\sqrt{r}} - r g_i g_k.
$$

Substituting $r_k = \frac{2g_{kr}}{T}$, we have

$$
\frac{1}{2} T^2 r_{ik} \xi_k = T \sqrt{r} g_{ik} \xi_k - \sqrt{r} g_{il} P_l g_k \xi_k + 2 r g_i g_k \xi_i \xi_k \\
- \sqrt{r} g_i g_{jk} P_j \xi_i \xi_k + \frac{\sqrt{r}}{T} g_{ji} P_l P_j g_i g_k \xi_k - r g_i g_k \xi_i \xi_k.
$$

Let $S = g_i \xi_i$. Then, by using $g_{ik} = g_{ki}$, we have

$$
r_{ik} \xi_i \xi_k = 2 \frac{\sqrt{r}}{T} g_{ik} (\xi_i - \frac{S}{T} P_i) (\xi_k - \frac{S}{T} P_k) + 2 r \frac{S^2}{T^2}.
$$

We have for some $M_1, M_2$ and $M_3$, which depend only on $c$,

$$
|g_i|, |g_{ik}| \leq M_3
$$

and

$$
0 < M_1 \leq \frac{T}{\sqrt{r}} \leq M_2
$$

for all $P \in \mathbb{R}^n \setminus \{0\}$. Thus we have

$$
r_{ik} \xi_i \xi_k \leq \nu_2 |\xi|^2
$$

for all $P, \xi \in \mathbb{R}^n$ where $\nu_2$ depends on $c$.

Now we define $h(\xi)$ and $\overline{h}(\xi)$ by

$$
h(\xi) = r_{ik} \xi_i \xi_k = |\xi|^2 \overline{h}(\xi).
$$

Since $g$ is strictly convex,

$$
\overline{h} \geq \frac{2}{M_2} \nu |\frac{\xi}{|\xi|} - \frac{S}{T} \frac{P}{|\xi|}|^2 + \frac{2}{M_2^2} \left( \frac{S}{|\xi|} \right)^2.
$$
Let \( m = \max_{\rho(P) = c} | P | \). Then
\[
\frac{| SP |}{| \xi | T} \leq \frac{Sm}{| \xi | M_1}
\]
for all \( P \in R^n \setminus \{0\} \). If \( \frac{S}{| \xi |} \geq \frac{M_1}{2m} \), then
\[
\overline{h} \geq \frac{M_1^2}{2m^2M_2^2}.
\]
On the other hand, if \( \frac{S}{| \xi |} \leq \frac{M_1}{2m} \), then
\[
\overline{h} \geq \frac{\nu}{M_2}.
\]
Thus
\[
h(\xi) \geq \nu_1 | \xi |^2,
\]
where \( \nu_1 = \min(\frac{M_1^2}{2m^2}, \frac{\nu}{M_2}) \), which depends only on \( \nu \) and \( c \).

Now we approximate \( f \). Let \( \psi : R \to R \) be \( C^\infty \) such that
\[
\psi(t) = 1 \quad \text{for } t \in (-\infty, 0]
\]
and
\[
\psi(t) = 0 \quad \text{for } t \in [1, \infty).
\]
Define \( r(P) \) by
\[
f\left( \frac{P}{\sqrt{r(P)}} \right) = \rho + \frac{\delta}{2}.
\]
From the definition of \( E_{\rho+\frac{\delta}{2}} \) and \( r(P) \), we see that
\[
\partial E_{\rho+\frac{\delta}{2}} = \{ P : r(P) = 1 \}
\]
and \( r(P) \in C^2 \). Let \( b(P) = (r(P) - 1)^+ \). Then
\[
b(P) = 0
\]
if \( P \in E_{\rho^+_{\frac{1}{2}}} \) and

\[
b(P) = r(P) - 1
\]

if \( P \in R^n \setminus E_{\rho^+_{\frac{1}{2}}} \).

Now we regularize \( b(P) \) with \( \epsilon' \) as (4) and get \( b_\epsilon'(P) \in C^\infty \). Then we see that \( b_\epsilon'(P) \) satisfies the same growth condition as \( r(P) \) if \( |P| \) is large enough. So \( b_\epsilon'(P) \) grows quadratically. Moreover \( b_\epsilon'(P) = 0 \) for \( P \in E_\rho \) and

\[
\nu_1 |\xi|^2 \leq b_\epsilon', P_1 P_2 (P) \xi_i \xi_j \leq \nu_2 |\xi|^2
\]

for all \( P \in R^n \setminus E_{\rho^++\delta} \) for some \( \nu_1 \) and \( \nu_2 \) if \( \epsilon' \) is small enough.

**THEOREM 4.** Suppose that \( f^\rho \) is defined by

\[
f^\rho(P) = \psi\left(\frac{f(P) - \rho - \delta}{\delta}\right)f(P) + \mu b_\epsilon'(P)
\]

for all \( P \in R^n \) and for some \( \mu > 0 \). If \( \mu \) is sufficiently large, then \( f^\rho \) satisfies the following ellipticity condition

\[
\lambda_1 |\xi|^2 \leq f^\rho_{P_1 P_2}(P) \xi_i \xi_j \leq \lambda_2 |\xi|^2
\]

for all \( P, \xi \in R^n \) and for some \( \lambda_1, \lambda_2 > 0 \) and \( f^\rho(P) = f(P) \) for \( P \in E_\rho \).

**proof.** We note that \( b_\epsilon'(P) = 0 \) and \( \psi(\frac{f(P) - \rho - \delta}{\delta}) = 1 \) for \( P \in E_\rho \). So \( f^\rho(P) = f(P) \) for \( P \in E_\rho \). By differentiating \( f^\rho \) with respect to \( P_i \), we have

\[
f^\rho_{P_i} = \frac{1}{\delta} \psi_{tt} f f P_i + \psi f P_i (P) + \mu b_\epsilon', P_i (P)
\]

and

\[
f^\rho_{P_1 P_2} = \frac{1}{\delta^2} \psi_{tt} f f P_1 f P_2 + \frac{1}{\delta} \psi_{tt} f P_1 P_2 + \frac{2}{\delta} \psi f P_i f P_i + \psi f P_1 P_2 + \mu b_\epsilon', P_1 P_2 (P).
\]

Since \( b_\epsilon' \) is convex,

\[
b_\epsilon', P_1 P_2 \xi_i \xi_j \geq 0
\]

for all \( P \in R^n \). Let \( P \in E_{\rho^++\delta} \). Then \( \psi = 1, \psi_t = 0 \) and \( \psi_{tt} = 0 \). Hence we have

\[
f^\rho_{P_1 P_2}(P) \xi_i \xi_j = f P_1 P_2 \xi_i \xi_j + \mu b_\epsilon', P_1 P_2 \xi_i \xi_j \geq \lambda |\xi|^2
\]
for all $\xi \in \mathbb{R}^n$. Since $|f_{P_1P_2}(P)| \leq M$ for some $M$ and for all $P \in E_{\rho+\delta}$, we have

$$f_{P_1P_2}\xi_i\xi_j \leq C |\xi|^2$$

for all $\xi \in \mathbb{R}^n$, where $C$ depends on $\mu$ and $M$.

Let $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$. Then we have

$$\nu_1 |\xi|^2 \leq b'_{\rho+\delta}P_1P_2\xi_i\xi_j \leq \nu_2 |\xi|^2$$

for all $\xi \in \mathbb{R}^n$. If $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$, then

$$|(\psi f)_{P_1P_2}| \leq M$$

for some $M$. Hence if $\mu$ is large enough, we have that for all $P \in E_{\rho+2\delta} \setminus E_{\rho+\delta}$

$$\mu b'_{\rho+\delta}P_1P_2\xi_i\xi_j + (\psi f)_{P_1P_2}\xi_i\xi_j \geq C_2 |\xi|^2$$

for some $C_2$ which depends on $M$ and $\mu$.

Since $\mu b'_{\rho+\delta}P_1P_2\xi_i\xi_j \leq \mu \nu_2 |\xi|^2$ and $|(\psi f)_{P_1P_2}| \leq M$ for all $P \in E_{\rho+\delta} \setminus E_{\rho}$, we have

$$\mu b'_{\rho+\delta}P_1P_2\xi_i\xi_j + (\psi f)_{P_1P_2}\xi_i\xi_j \leq C_3 |\xi|^2$$

where $C_3$ depends on $M$ and $\mu$.

If $P \in \mathbb{R}^n \setminus E_{\rho+2\delta}$, then $\psi = 0$ and $f^\rho(P) = \mu b'_{\rho}(P)$. Hence by Theorem 3 we have

$$f^\rho_{P_1P_2}\xi_i\xi_j \leq \mu \nu_2 |\xi|^2$$

for all $P, \xi \in \mathbb{R}^n$.

4. Regularity

By using a maximum principle and existence theorem for quasilinear elliptic equations due to P. Hartman and G. Stampaccia [3] we obtain $C^{1,\alpha}(\bar{\Omega})$ regularity for a minimizer if $(u_0, \partial\Omega)$ satisfies a certain bounded slope condition.
THEOREM 5. Suppose that $\Omega$ is a bounded open connected subset of $\mathbb{R}^n$ with $\partial \Omega \in C^{1,1}$. Moreover suppose that $u_0$ satisfies the following "bounded slope condition": there exists a constant $M$ such that for each point $x_0 \in \partial \Omega$, there exist linear functions $\pi_{x_0}^\pm$ such that

$$f(D\pi_{x_0}^\pm) \leq M$$

and

$$\pi_{x_0}^-(x-x_0) \leq u_0(x) - u_0(x_0) \leq \pi_{x_0}^+(x-x_0)$$

for all $x \in \partial \Omega$. Then the minimizer $u$ with respect $K$ is $C^{1,\alpha}(\Omega)$ for all $0 \leq \alpha < 1$.

J. Moser observed in [4] that if $v$ is a solution of a linear elliptic equation

$$(10) \quad D_i(a_{ij}(x)D_ju) = 0$$

with $a_{ij}$ measurable and

$$c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq c_1|\xi|^2$$

for some positive constants $c_0$ and $c_1$, then for any convex function $h$, $h(u)$ is a subsolution of (10). We prove a similar theorem for the derivatives of solutions of quasilinear elliptic equations.

THEOREM 6. Suppose that $A_i \in C^1(\mathbb{R}^n)$ satisfies the following ellipticity condition:

$$c_0|\xi|^2 \leq A_i(P)\xi_i\xi_j \leq c_1|\xi|^2$$

for all $P, \xi \in \mathbb{R}^n$ and for some positive constants $c_0$ and $c_1$. Moreover suppose that $g : R \to R$ is nonincreasing and in $C^1$.

Let $v \in W^{1,2}(\Omega)$ be a solution to the quasilinear equation

$$(12) \quad D_i(A_i(Dv)) + g(v) = 0.$$

Suppose that $G \in C^2(\mathbb{R}^n; R)$ is a convex function with $G(0) = 0$ as minimum. Then $G(Dv)$ is a subsolution to

$$(13) \quad D_i(A_{i,P}(Dv)D_jw) = 0.$$
proof. First we prove that \( v \in W^{2,2}_{\text{loc}}(\Omega) \) by difference quotient argument.

Let \( \Omega' \subset \subset \Omega \) and \( d < \text{dist}(\Omega', \partial \Omega) \). Let \( h \leq \frac{1}{4}d \) and \( e_k \) be \( k \)-th direction unit coordinate vector for \( k = 1, \ldots, n \). Let \( \psi \in C^\infty(\Omega) \), \( |D\psi| \leq \frac{c}{d} \) for some \( c \) and \( \text{supp}(\psi) \pm \frac{1}{4}de_k \subset \Omega \). We apply \((v(x + he_k) - v(x))\psi^2(x)\) as a test function to (12). Hence we have that

\[
\int_{\Omega} [A_i(Dv(x + he_k)) - A_i(Dv(x))]D_i[(v(x + he_k) - v(x))\psi^2(x)]dx
\]

\[
- \int_{\Omega} [g(v(x + he_k)) - g(v(x))][v(x + he_k) - v(x)]\psi^2(x)dx = 0
\]

for all \( k = 1, \ldots, n \). Since \( g \) is nonincreasing, we have that

\[
[g(v(x + he_k)) - g(v(x))][v(x + he_k) - v(x)]\psi^2(x) \leq 0
\]

for all \( x \in \Omega \) and \( k = 1, \ldots, n \). By using the ellipticity of \( A_i \) and the equation we have

\[
\frac{c_0}{h^2} \int_{\Omega} |Dv(x + he_k) - Dv(x)|^2 \psi^2 dx
\]

\[
\leq \frac{1}{h^2} \int_{\Omega} [A_i(Dv(x + he_k)) - A_i(Dv(x))][D_i v(x + he_k) - D_i v(x)] \psi^2 dx
\]

\[
\leq - \frac{2}{h^2} \int_{\Omega} [A_i(Dv(x + he_k)) - A_i(Dv(x))] [v(x + he_k) - v(x)]D_i \psi(x) \psi(x)dx
\]

\[
\leq 2c_1 \int_{\Omega} \left| \frac{Dv(x + he_k) - Dv(x)}{h} \right| \left| \frac{v(x + he_k) - v(x)}{h} \right| |D\psi(x)| \psi(x) dx.
\]

Now by using Holder's inequality on the right hand side of the last inequality, we have

\[
\int_{\Omega'} \left| \frac{Dv(x + he_k) - Dv(x)}{h} \right|^2 dx \leq \frac{c}{d^2} \int_{\Omega''} \left| \frac{v(x + he_k) - v(x)}{h} \right|^2 dx
\]

for some \( \Omega' \subset \subset \Omega'' \subset \Omega \), for all \( 0 < h < \frac{1}{4}d \) and \( k \). So \( v \in W^{2,2}_{\text{loc}}(\Omega) \) and we can differentiate formally with respect to \( x_k \) to obtain

\[
D_i(A_i, D_j(Dv)) + g'(v)D_k v = 0
\]
for each $k$. Let $\eta$ be a nonnegative $C_0^\infty(\Omega)$ function. Then

$$\int_\Omega A_{i,j}(Dv)D_j G(Dv)D_i \eta \, dx = \int_\Omega A_{i,j}(Dv)G_{P_k}(Dv)D_j D_k v D_i \eta \, dx.$$ 

Since

$$D_i(G_{P_k}(Dv)\eta) = G_{P_k}P_i D_i D_l v \eta + G_{P_k}(Dv) D_l \eta$$

and $G_{P_k}(Dv) \eta \in W^{1,2}_0(\Omega)$, we have

$$\int_\Omega A_{i,j} G_{P_k}(Dv)D_j D_k v D_i \eta \, dx = \int_\Omega A_{i,j} D_j D_k v D_i (G_{P_k}(Dv) \eta) \, dx - \int_\Omega A_{i,j} G_{P_k}P_i (Dv) D_l D_k v D_i \eta \, dx$$

Since $G$ is radially increasing and $g' \leq 0$,

$$\int_\Omega g'(v)D_k v G_{P_k}(Dv) \eta \, dx \leq 0.$$ 

Since $A_{i,j}$ and $G_{P_k}P_i$ are positive definite matrices,

$$A_{i,j} G_{P_k}P_i (Dv) D_j D_k v D_i \eta \geq 0.$$ 

Therefore we have

$$\int_\Omega A_{i,j}(Dv)D_j G(Dv)D_l \eta \, dx \leq 0$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. Hence $G(Dv)$ is a subsolution to

$$D_i(A_{i,j} D_j w) = 0$$

and this completes the proof.

We have the following lemma for the solutions of homogeneous equations.
LEMMA 3. Let \( v \in W^{1,2}(\Omega) \) be a solution to
\[
D_i(A_i(Dv)) = 0
\]
where \( A_i : R^n \rightarrow R \) satisfies the ellipticity condition (12). Let \( G : R^n \rightarrow R \) be convex and in \( C^1 \). Then \( G(Dv) \) is a subsolution to
\[
D_i(A_i,p_j(Dv)D_jw) = 0.
\]

Since \( G(Dv) \) is a subsolution of a linear elliptic equation, we have a maximum principle.

LEMMA 4. Let \( G \) and \( v \in C^1(\Omega) \) satisfy the same conditions as in Theorem 6. Then we have the following maximum principle
\[
(14) \quad \max_{\Omega} G(Dv) \leq \max_{\partial \Omega} G(Dv).
\]

**proof.** Let \( M = \max_{\partial \Omega} G(Dv) \) and \( w = (G(Dv) - M - \epsilon)^+ \) for some \( \epsilon > 0 \). Then we see \( w \in W_0^{1,2} \). So by using \( w \) as a test function to (13), we have
\[
\int_{\{x : M + \epsilon \leq G(Dv(x))\}} A_i,p_j D_i(G(Dv))D_j(G(Dv))dx \leq 0
\]
and
\[
\int_{\{x \in \Omega : G(Dv(x)) \geq M + \epsilon\}} |D(G(Dv))|^2 dx = 0.
\]
By using Sobolev inequality we have \( \text{meas}\{x \in \Omega : M + \epsilon \leq G(Dv(x))\} = 0 \) for all \( \epsilon > 0 \).

Now we prove Theorem 5 by using monotone operator theory as in [3].

**proof of Theorem 5.** Let \( f^\rho \) be the approximation of \( f \) in the theorem 4 such that \( f^\rho(P) = f(P) \) for all \( P \in \{ P : f(P) \leq \rho \} \cup \{ P : f^\rho(P) \leq \rho \} \) and let \( f^\rho \) satisfy the quadratic growth condition. Let \( u^\rho \) be the minimizer of
\[
I^\rho[u^\rho] = \int_{\Omega} f^\rho(Du^\rho)dx
\]
with respect to $K^p = \{ v \in W^{1,2} : v - u_0 \in W^{1,2}_0 \}$.

From section 2.1, we know that there exists a unique minimizer $u^p$ for each $p$. Fix $L \geq 2M$, where $M$ is the constant defined in the bounded slope condition of Theorem 5. We know that $u^L$ satisfies the Euler-Lagrange equation

\[(15) \quad D_t(\mathcal{F}(Du^L)) = 0 \]

with $u^L - u_0 \in W^{1,2}_0$. We see that $(u_0, \partial \Omega)$ has the ordinary bounded slope condition and

\[|D\pi^\pm_{x_0}| \leq C \]

for all $x \in \partial \Omega$, where $C$ is independent of $x_0$.

Since $u_0$ is Lipschitz and $(u_0, \partial \Omega)$ satisfies the bounded slope condition, there exists a $u^L \in C^{1,\alpha}(\Omega)$ for all $0 \leq \alpha < 1$ which satisfies the Euler-Lagrange equation (15) by the Theorem 13.1 and 14.1 in [3]. By the uniqueness, $u^L = u^L$. Since $f^L$ is convex, from the maximum principle (Lemma 4), we see that

\[
\max_{\Omega} f^L(Du^L) \leq \max_{\partial \Omega} f^L(Du^L).
\]

Since $u^L = u_0$ on $\partial \Omega$ and $\pi^-_{x_0}(x) \leq u^L(x) \leq \pi^+_{x_0}(x)$ for all $x \in \Omega$,

\[
\frac{\partial}{\partial \eta} u^L(x_0) = \frac{\partial}{\partial \eta} u_0(x_0)
\]

for all tangent vector $\eta$ to $\partial \Omega$ at $x_0$ and

\[
\frac{\partial}{\partial \tau} \pi^+_{x_0} \leq \frac{\partial}{\partial \tau} u^L(x_0) \leq \frac{\partial}{\partial \tau} \pi^-_{x_0}
\]

for all outward normal vector $\tau$ to $\partial \Omega$ at $x_0$. So we see that

\[Du^L(x_0) = tD\pi^+_{x_0} + (1 - t)D\pi^-_{x_0}\]

for some $0 \leq t \leq 1$ and

\[f^L(Du^L(x_0)) \leq tf^L(D\pi^+_{x_0}) + (1 - t)f^L(D\pi^-_{x_0}) \leq M\]

for all $x_0 \in \partial \Omega$. So

\[
\max_{\Omega} f^L(Du^L) \leq M.
\]

Since $f^L(P) = f(P)$ if $f^L(P) \leq L$, we conclude that $f(Du^L) = f^L(Du^L)$ for all $x \in \Omega$ and hence $u^L \in K$. From the uniqueness of the minimizer, $u^L = u$ and $u$ is $C^{1,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha < 1$. 
5. Counterexamples

In this section we construct some counterexamples which exhibit that if the boundary data do not satisfy the bounded slope condition, then a minimizer may not have a continuous derivative.

Let \( 0 < \theta < 1 \) and \( O = (-1, 1) \times (-1, 1) \).

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function such that

\[
f(P) = (1 - P_1^2)^{-\theta} + (1 - P_2^2)^{-\theta}\]

for all \( P \in O \) and

\[
f(P) = \infty
\]

for \( P \in \mathbb{R}^2 \setminus O \).

By direct computation, we have

\[
f_{P_1}(P) = 2\theta P_1(1 - P_1^2)^{-\theta-1},
\]

\[
f_{P_2}(P) = 2\theta P_2(1 - P_2^2)^{-\theta-1},
\]

\[
f_{P_1P_1}(P) = 2\theta(1 - P_1^2)^{-\theta-2}(1 + (2\theta + 1)P_1^2),
\]

\[
f_{P_1P_2}(P) = 0,
\]

\[
f_{P_2P_2}(P) = 2\theta(1 - P_2^2)^{-\theta-2}(1 + (2\theta + 1)P_2^2)
\]

and we can see

\[
f_{P_1P_2}(P)\xi_i\xi_j \geq 2\theta|\xi|^2
\]

for all \( P \in O \) and \( \xi \in \mathbb{R}^2 \).

Suppose \( \Omega = (0, \frac{1}{4}) \times (-1, 1) \), \( \Omega_1 = (0, \frac{1}{4}) \times (0, 1) \) and \( \Omega_2 = (0, \frac{1}{4}) \times (-1, 0) \).

We define \( I[v] \) by

\[
I[v] = \int_{-1}^{1} \int_0^{\frac{1}{4}} f(Dv)dx\,dy
\]

for all \( v \in W^{1,\infty} \). Let \( u_1(x, y) = x(1 - y) \) in \( \overline{\Omega}_1 = [0, \frac{1}{4}] \times [0, 1] \).

Now reflect \( u_1 \) with respect to \( x \) axis and set \( u_2 = x(1 + y) \) in \( \overline{\Omega}_2 = [0, \frac{1}{4}] \times [-1, 0] \). Define \( u = u_1 \) in \( \overline{\Omega}_1 \) and \( u = u_2 \) in \( \overline{\Omega}_2 \).
LEMMA 5. Let \( w \) be any admissible function for \( I( \text{i.e., } w = u \text{ on } \partial \Omega) \) and \( I[w] < \infty \). Then

\[
  w(x, 0) = x
\]

for all \( 0 \leq x \leq \frac{1}{4} \).

**proof.** We prove by contradiction. First we note that \( w \) is a Lipschitz function. Suppose that the lemma is false and we assume that \( w(x_0, 0) > x_0 \) for some \( x_0 \), where \( 0 < x_0 < \frac{1}{4} \). Define \( \delta = w(x_0, 0) - x_0 \). We regularize \( w \) with \( \epsilon \) as (4). Then \( w_\epsilon \to w \) uniformly for all \( x \in \Omega' \subset \subset \Omega \) and by Jensen’s inequality,

\[
  f(Dw_\epsilon) \leq f(Dw)_\epsilon < \infty
\]

for all \( x \in \Omega' \) if \( \epsilon \) is sufficiently small.

Let \( \delta_1 > 0 \) be so small that

\[
  w(\delta_1, 0) < \frac{\delta}{5}
\]

and let \( \epsilon \) be so small that

\[
  | w_\epsilon(\delta_1, 0) - w(\delta_1, 0) | \leq \frac{\delta}{5}
\]

and

\[
  | w_\epsilon(x_0, 0) - w(x_0, 0) | \leq \frac{\delta}{5}.
\]

Then we see that

\[
  \frac{w_\epsilon(x_0, 0) - w_\epsilon(\delta_1, 0)}{x_0 - \delta_1} \geq 1 + \delta_2
\]

for some \( \delta_2 > 0 \) independently for all small \( \epsilon \). So for some \( \delta_1 \leq x_1 \leq x_0 \)

\[
  \frac{\partial w_\epsilon}{\partial x}(x_1, 0) > 1
\]

and

\[
  f(Dw_\epsilon(x_1, 0)) = \infty.
\]

This contradicts the fact that

\[
  f(Dw_\epsilon) < \infty
\]

for all \( x \in \Omega' \subset \subset \Omega \).
THEOREM 7. $u$ is a minimizer and $Du$ is not continuous.

proof. By direct computation, we see that $I[u] < \infty$ and $u$ is an admissible function. Moreover for all $\psi \in C_0^\infty(\Omega_1)$,

$$
\int_{\Omega_1} f_{P_1}(Du) \frac{\partial \psi}{\partial x} + f_{P_2}(Du) \frac{\partial \psi}{\partial y} dxdy
$$

$$
= - \int_{\Omega_1} f_{P_1}(Du) \frac{\partial^2 u}{\partial x \partial x} \psi + 2f_{P_1}P_2(Du) \frac{\partial^2 u}{\partial x \partial y} \psi dxdy
$$

$$
+ f_{P_2}P_2(Du) \frac{\partial^2 u}{\partial y \partial y} \psi dxdy.
$$

We have, by direct computation,

$$
\frac{\partial^2 u}{\partial x \partial x} = \frac{\partial^2 u}{\partial y \partial y} = 0
$$

and

$$
f_{P_1}P_2 = 0.
$$

We see that $u$ satisfies the Euler-Lagrange equation in $\Omega_1$. Similarly we see that $u$ satisfies the Euler-Lagrange equation in $\Omega_2$. Since every admissible function must have the same data on the line $y = 0$, we conclude that $u$ is a minimizer.

Since

$$
\frac{\partial u}{\partial y} = -x
$$

in $\Omega_1$ and

$$
\frac{\partial u}{\partial y} = x
$$

in $\Omega_2$, $Du$ is not continuous on the line $y = 0$.

REMARK. We note that the minimizers do not have the unique continuation property.
References


Department of Mathematics
POSTECH,
Pohang 790-330, Korea