THE DIRICHLET EIGENVALUE ESTIMATE ON A COMPACT RIEMANNIAN MANIFOLD

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§1. Introduction

Let $M$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$. We consider the following Dirichlet eigenvalue problem on $M$ of the equation

\[
\begin{align*}
\Delta u &= -\lambda u \text{ in } M \\
u &\equiv 0 \text{ on } \partial M.
\end{align*}
\]

It is well known that the set of eigenvalues $\{\lambda_k\}$ of (1.1) can be arranged in a nondecreasing order as follows:

\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots
\]

and $\lambda_1 \int u^2 \leq \int |\nabla u|^2$ for all $u \in H^1_0(M)$. For application, it is important to estimate the lower bound of $\lambda_1$. In case that $M$ is a submanifold of some manifold $N$, the Ricci curvature of $N$ has the positive lower bound in $M$ and the average curvature of $\partial M$ is nonnegative, Sperb showed a lower bound of $\lambda_1$ in [2]. For compact Riemannian manifold with nonconvex boundary, the first Neumann eigenvalue is estimated by R. Chen[1]. Our purpose in this paper is to estimate the lower bound of $\lambda_1$ on compact Riemannian manifold with nonconvex boundary.

DEFINITION. Let $\partial M$ be the boundary of a compact Riemannian manifold $M$. Then $\partial M$ satisfies the "interior rolling $\varepsilon$-ball" condition if for each point $p \in \partial M$, there is a geodesic ball $B_q(\frac{\varepsilon}{2})$, centered at $q \in M$ with radius $\frac{\varepsilon}{2}$, such that $p = \overline{B_q(\varepsilon/2)} \cap \partial M$ and $B_q(\frac{\varepsilon}{2}) \subset M$.

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Theorem 1.1. Let $M$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial M$. Let $\partial M$ satisfy the "interior rolling $\epsilon$-ball" condition. Let $R, K$ and $H$ be nonnegative constants such that the Ricci curvature of $M$ is bounded below by $-R$, the mean curvature of $\partial M$ is bounded below by $-K$ and the second fundamental form elements of $\partial M$ is bounded below by $-H$. If $u$ is a solution of the equation

\begin{equation}
\Delta u + \lambda_1 u = \text{in } M \\
u \equiv 0 \text{ on } \partial M,
\end{equation}

where $\lambda_1$ is the first Dirichlet eigenvalue. Then

$$\lambda_1 \geq \frac{1}{\sqrt{1 + 2(n-1)\varepsilon K}} \frac{(1 - \alpha^2)}{(n-1)\rho^2} (1 + B)\exp(-(1 + B)),$$

where

$$B = \left[1 + \frac{(n-1)\rho^2}{1 - \alpha^2} C\right]^{\frac{1}{2}},$$

$$C = \frac{(2n-3)^2 + \alpha^2(10n-11)}{\alpha^2} (n-1)K^2 + 2\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} R$$

$$+ (n-1)K\left\{\frac{1}{\varepsilon} + 2(n-1)(1 + 3H)\right\}, \quad 0 < \alpha \leq \frac{1}{2},$$

and $\rho$ is the radius of the largest geodesic ball contained in $M$ and the upper bound of $\varepsilon$ is given by (2.11) and (2.12).

In §2, we shall give a gradient estimate which is essential in proof of Theorem 1.1. In §3, we shall give a proof of Theorem 1.1.

§2 A gradient estimate

Lemma 2.1.[2]. Let $S^{n-1}$ be a hypersurface of the Riemannian manifold $M$ and $\Delta_s$ the Laplacian in the induced metric of $S^{n-1}$. At any point of $S^{n-1}$, the following relation holds:

$$\Delta u = \Delta_s u + (n-1)K_0 \frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2},$$

where $K_0$ is the mean curvature, $\nu$ is the outward normal vector.
THEOREM 2.2. Let $M$ and $\partial M$ satisfy the hypothesis of Theorem 1.1. If $u$ is a solution of the equation (1.2), then

$$\frac{|\nabla u|^2}{(\beta - u)^2}(x) \leq \frac{(n - 1)}{(1 - \alpha^2)} \left[ \frac{(2n - 3)^2 + \alpha^2 (10n - 11)}{\alpha^2} (n - 1)K^2 
+ 2\{1 + 2(n - 1)\varepsilon K\}^{\frac{1}{2}} R 
+ \frac{2\beta}{\beta - \sup u} \{1 + 2(n - 1)\varepsilon K\}^{\frac{1}{2}} \lambda_1 + C_1 \right]$$

where $C_1 = (n - 1)K \{\frac{1}{\varepsilon} + 2(n - 1)(1 + 3H)\}$, and $\beta > \sup u$.

Proof. Let $\psi(r)$ be a nonnegative $C^2$-function defined on $[0, \infty)$ such that,

$$\psi(r) = \begin{cases} 
2(n - 1)\varepsilon K & \text{if } r \in [0, \frac{\varepsilon}{2}) \\
2(n - 1)\varepsilon K & \text{if } r \in [\varepsilon, \infty) 
\end{cases}$$

with $\psi(0) = 0$, $\psi'(0) = 4(n - 1)K$, $0 \leq \psi'(r) \leq 4(n - 1)K$ and $\psi'' \geq -\frac{2(n - 1)K}{\varepsilon}$. Define $\phi(x) = \psi(r(x))$, where $r(x)$ denotes the distance function from boundary $\partial M$ to $x \in M$. For $\beta > 1 = \sup u$, we define the function

$$G(x) = (1 + \phi)^{\frac{1}{2}} \frac{|\nabla u|^2}{(\beta - u)^2} \text{ on } M.$$ 

By the compactness of $M$, there is a point $x_0 \in M$ such that $G$ achieves its maximum. Suppose that $x_0$ is a boundary point of $\partial M$. At $x_0$, we may choose an orthonormal frame field $e_1, e_2, \ldots, e_n$ such that $e_n = \frac{\partial}{\partial \nu}$, where $\frac{\partial}{\partial \nu}$ is the unit outward normal vector. Then we have

$$\frac{\partial G}{\partial \nu}(x_0) = \frac{1}{2} \frac{\partial \phi}{\partial \nu} \frac{|\nabla u|^2}{(\beta - u)^2} + \frac{2u}{\partial \nu} \frac{2|\nabla u|^2}{\partial \nu} \frac{1}{(\beta - u)^2} + \frac{2|\nabla u|^2}{(\beta - u)^3} \frac{\partial u}{\partial \nu} \geq 0.$$ 

From (1.2), it is clear that

$$\Delta u(x_0) = -\lambda_1 u(x_0) = 0, \quad \Delta_{\partial M} u(x_0) = 0.$$
Let $K_0$ be the mean curvature at $x_0$. By Lemma 2.1, at $x_0$,

$$\Delta u = \Delta_{\partial M} u + (n - 1) K_0 \frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial \nu^2}.$$

Hence, we have

$$(2.2) \quad \frac{\partial^2 u}{\partial \nu^2} = -(n - 1) K_0 \frac{\partial u}{\partial \nu} = (n - 1) K_0 |\nabla u|, \quad \text{at } x_0 \in \partial M.$$

Using (2.2), we obtain that

$$\frac{\partial G}{\partial \nu}(x_0) = \frac{1}{2} \frac{\partial \phi}{\partial \nu} \frac{|\nabla u|^2}{(\beta - u)^2} - 2(n - 1) K_0 \frac{|\nabla u|^2}{(\beta - u)^2} - 2 \frac{|\nabla u|^3}{(\beta - u)^3}$$

$$\leq \frac{|\nabla u|^2}{(\beta - u)^2} \left( \frac{1}{2} \frac{\partial \phi}{\partial \nu} + 2(n - 1) K - 2 \frac{|\nabla u|}{(\beta - u)} \right)$$

$$= \frac{|\nabla u|^2}{(\beta - u)^2} \left( -2(n - 1) K + 2(n - 1) K - 2 \frac{|\nabla u|}{(\beta - u)} \right) < 0.$$

This contradicts (2.1). Therefore $x_0$ is an interior point of $M$ and hence, $\nabla G(x_0) = 0$ and $\Delta G(x_0) \leq 0$. At $x_0$, we may choose an orthonormal frame field $\{e_i\}$ such that $u_1(x_0) = |\nabla u(x_0)|$. Since, for each $i$, $G_i(x_0) = 0$, we obtain that

$$(2.3) \quad \text{if } i \neq 1, \quad u_{1i} = -\frac{1}{4}(1 + \phi)^{-1} \phi_i |\nabla u|,$$

$$\text{if } i = 1, \quad u_{11} = -\frac{|\nabla u|^2}{(\beta - u)} - \frac{1}{4}(1 + \phi)^{-1} \phi_1 |\nabla u|.$$

It is clear that

$$(2.4) \quad \sum_{i,j=1}^{n} (u_{ji})^2 \geq \sum_{i=1}^{n} u_{ii}^2 \geq u_{11}^2 + \frac{1}{n - 1} (\Delta u - u_{11})^2$$

$$\geq u_{11}^2 + \frac{u_{11}^2}{2(n - 1)} - \frac{(\Delta u)^2}{n - 1}$$

$$= \frac{2n - 1}{2(n - 1)} \left( \frac{|\nabla u|^4}{(\beta - u)^2} + \frac{1}{2} (1 + \phi)^{-1} \phi_1 \frac{|\nabla u|^3}{(\beta - u)} \right.$$

$$+ \left. \frac{1}{16} (1 + \phi)^{-2} \phi_1^2 |\nabla u|^2 \right) - \frac{1}{n - 1} \lambda_1^2 u_1^2.$$
By using (2.3) and $u_{ijk} - u_{ikj} = \sum_{l=1}^{n} u_{l}R_{lijk}$, we have

(2.5)

$$\Delta G(x_0) = -\frac{1}{4}(1 + \phi)^{-\frac{3}{2}} |\nabla \phi|^2 \frac{|\nabla u|^2}{(\beta - u)^2} + \frac{1}{2}(1 + \phi)^{-\frac{1}{2}} \Delta \phi \frac{|\nabla u|^2}{(\beta - u)^2}$$

$$+ 2(1 + \phi)^{-\frac{1}{2}} \phi \frac{u_1}{(\beta - u)^2} \left(-\frac{|\nabla u|^2}{(\beta - u)} - \frac{1}{4}(1 + \phi)^{-1} \phi_1 |\nabla u|\right)$$

$$+ \sum_{i=2}^{n} 2(1 + \phi)^{-\frac{1}{2}} \phi \frac{u_1}{(\beta - u)^2} \left(-\frac{1}{4}(1 + \phi)^{-1} \phi_i |\nabla u|\right)$$

$$+ 2(1 + \phi)^{-\frac{1}{2}} \phi_1 \frac{|\nabla u|^3}{(\beta - u)^3} + \sum_{i,j=1}^{n} 2(1 + \phi)^{\frac{1}{2}} \frac{(u_{jj})^2}{(\beta - u)^2}$$

$$+ \sum_{j=1}^{n} 2(1 + \phi)^{\frac{1}{2}} \frac{(u_{jj})(\Delta u)_j}{(\beta - u)^2} + \sum_{i,j=1}^{n} 2(1 + \phi)^{\frac{1}{2}} \frac{u_j u_i R_{ij}}{(\beta - u)^2}$$

$$+ \frac{8(1 + \phi)^{\frac{1}{2}}}{(\beta - u)^3} |\nabla u|^2 \left(-\frac{|\nabla u|^2}{(\beta - u)} - \frac{1}{4}(1 + \phi)^{-1} \phi_1 |\nabla u|\right)$$

$$+ 2(1 + \phi)^{\frac{1}{2}} \left(-\lambda_1 u |\nabla u|^2\right) + 6(1 + \phi)^{\frac{1}{2}} \frac{|\nabla u|^4}{(\beta - u)^3}.$$ 

Multiplying (2.5) by $(1 + \phi)^{\frac{1}{2}} \frac{(\beta - u)^2}{|\nabla u|^2}$ and substituting (2.4), we have

(2.6)

$$0 \geq \frac{(1 + \phi)}{(n - 1)} \frac{|\nabla u|^2}{(\beta - u)^2} + \frac{-(2n - 3)}{2(n - 1)} \phi_1 \frac{|\nabla u|}{(\beta - u)}$$

$$- \frac{2(1 + \phi)}{(n - 1)} \frac{\lambda_1^2 u^2}{|\nabla u|^2} + \frac{2(n - 1)}{16(n - 1)} (1 + \phi)^{-1} \phi_1^2$$

$$- \frac{2\lambda_1 u(1 + \phi)}{(\beta - u)} - 2R(1 + \phi) - 2\lambda_1(1 + \phi)$$

$$- \frac{3}{4}(1 + \phi)^{-1} |\nabla \phi|^2 + \frac{1}{2} \Delta \phi.$$

It is clear that

(2.7)

$$\frac{\alpha^2(1 + \phi)}{(n - 1)} \frac{|\nabla u|^2}{(\beta - u)^2} - \frac{(2n - 3)}{2(n - 1)} \phi_1 \frac{|\nabla u|}{(\beta - u)}$$

$$\geq -\frac{1}{16} \frac{(2n - 3)^2}{\alpha^2(n - 1)} (1 + \phi)^{-1} \phi_1^2.$$
Substituting (2.7) into (2.6), we have, for $0 < \alpha < 1$,

\[(2.8) \quad 0 \geq \frac{(1 - \alpha^2)(1 + \phi)|\nabla u|^2}{(n-1)(\beta-u)^2} \]
\[\quad - \frac{1}{16\alpha^2} \frac{(2n-3)^2}{(n-1)} (1 + \phi)^{-1} \phi_1^2 \]
\[\quad - \frac{2(1 + \phi)\lambda_1^2 u^2}{(n-1)|\nabla u|^2} + \frac{(2n-1)}{16(n-1)} (1 + \phi)^{-1} \phi_1^2 \]
\[\quad - \frac{2\lambda_1 u}{(\beta-u)} (1 + \phi) - 2(1 + \phi)(R + \lambda_1) \]
\[\quad - \frac{3}{4} (1 + \phi)^{-1} |\nabla \phi|^2 + \frac{1}{2} \Delta \phi. \]

Multiplying (2.8) by $\frac{|\nabla u|^2}{(\beta-u)^2}$, we obtain

\[(2.9) \quad 0 \geq \frac{1 - \alpha^2}{(n-1)} G(x_0)^2 - G(x_0) \left\{ \frac{(2n-3)^2}{16\alpha^2(n-1)} (1 + \phi)^{-\frac{3}{2}} \phi_1^2 \right\} \]
\[\quad + \frac{2\lambda_1 u}{(\beta-u)} (1 + \phi)^{\frac{1}{2}} - \frac{(2n-1)}{16(n-1)} (1 + \phi)^{-\frac{3}{2}} \phi_1^2 \]
\[\quad + 2(1 + \phi)^{\frac{1}{2}} (R + \lambda_1) + \frac{3}{4} (1 + \phi)^{-\frac{3}{2}} |\nabla \phi|^2 \]
\[\quad - \frac{1}{2} \Delta \phi (1 + \phi)^{-\frac{1}{2}} \right\} - \frac{2(1 + \phi)\lambda_1^2 u^2}{(\beta-u)^2(n-1)}. \]

From (2.9), we obtain that

\[(2.10) \quad 0 \geq \frac{(1 - \alpha^2)}{(n-1)} G(x_0)^2 \]
\[\quad - \left\{ \frac{(2n-3)^2 + \alpha^2(10n-11)}{16\alpha^2(n-1)} (1 + \phi)^{-\frac{3}{2}} |\nabla \phi|^2 \right\} \]
\[\quad + 2(1 + \phi)^{\frac{1}{2}} (R + \lambda_1) + \frac{2\lambda_1 u}{\beta-u} (1 + \phi)^{\frac{1}{2}} \]
\[\quad - \frac{1}{2} \Delta \phi (1 + \phi)^{-\frac{1}{2}} \} G(x_0) \]
\[\quad - \frac{2(1 + \phi)\lambda_1^2 u^2}{(\beta-u)^2(n-1)}, \quad \text{for } 0 < \alpha \leq \frac{1}{2}. \]
To compute $\Delta \phi$, let $\partial M(\epsilon)$ be the set $\{x \in M \mid r(x) \leq \epsilon\}$, and $k_\epsilon$ be the upper bound of the sectional curvature in $\partial M(\epsilon)$. We may choose $\epsilon$ to be small so that

(2.11) \[ \sqrt{k_\epsilon} \tan(\epsilon \sqrt{k_\epsilon}) \leq \frac{H}{2} + \frac{1}{2} \]

(2.12) \[ \frac{H}{\sqrt{k_\epsilon}} \tan(\epsilon \sqrt{k_\epsilon}) \leq \frac{1}{2}. \]

By using an indez comparison theorem in Riemannian geometry [3], one can show that if $x \in \partial M(\epsilon)$, we have

$$\Delta r \geq -(n - 1) \frac{H + \sqrt{k_\epsilon} \tan(\epsilon \sqrt{k_\epsilon})}{1 - \tan(\epsilon \sqrt{k_\epsilon})H/\sqrt{k_\epsilon}} \geq -(n - 1)(3H + 1).$$

Then we have

(2.13) \[ \Delta \phi = \psi''|\nabla r|^2 + \psi' \Delta r \]

$$\geq -\frac{2(n - 1)K}{\epsilon} - 4(n - 1)^2 K(3H + 1).$$

Let $C_1 = (n - 1)K\left(\frac{1}{\epsilon} + 2(n - 1)(1 + 3H)\right)$.

Substituting (2.13) into (2.12), (2.12) becomes

$$0 \geq \left(\frac{1 - \alpha^2}{n - 1}\right) G(x_0)^2 - \left\{\frac{(2n - 3)^2 + \alpha^2(10n - 11)}{\alpha^2}(n - 1)K^2 + 2(1 + 2(n - 1)\epsilon K)^{\frac{1}{2}} R \right. $$

$$+ \left. \frac{2\beta}{\beta - \sup u} (1 + 2(n - 1)\epsilon K)^{\frac{1}{2}} \lambda_1 + C_1 \right\} G(x_0) $$

$$- \frac{2(1 + 2(n - 1)\epsilon K) \lambda_1^2 (\sup u)^2}{(n - 1)(\beta - \sup u)^2}. \]

Hence

$$\frac{\|
abla u\|^2}{(\beta - u)^2}(x) \leq G(x_0) \leq \left(\frac{n - 1}{1 - \alpha^2}\right) \left[\frac{(2n - 3)^2 + \alpha^2(10n - 11)}{\alpha^2}(n - 1)K^2 + 2(1 + 2(n - 1)\epsilon K)^{\frac{1}{2}} R \right. $$

$$+ \left. \frac{2\beta}{\beta - \sup u} (1 + 2(n - 1)\epsilon K)^{\frac{1}{2}} \lambda_1 + C_1 \right].$$
REMARK. By the "interior rolling $\epsilon$-ball" condition, we can choose a geodesic from boundary to $x_0$ which has no focal point. Hence we can use the index comparison theorem.

§3. Proof of Theorem 1.1

Proof. We may assume that $0 \leq u \leq 1$. From Theorem 2.2, we know that

$$\frac{|\nabla u|}{(\beta - u)} \leq \left( \frac{n - 1}{1 - \alpha^2} \right)^{\frac{1}{2}} \left[ \frac{(2n - 3)^2 + \alpha^2 (10n - 11)}{\alpha^2} (n - 1)K^2 + 2\{1 + 2(n - 1)\epsilon K\}^{\frac{1}{2}} R \\
+ \frac{2\beta}{\beta - 1} \left\{ \frac{1}{\epsilon} + 2(n - 1)\epsilon K \right\}^{\frac{1}{2}} \lambda_1 + C_1 \right]^{\frac{1}{2}}.$$ 

Let $x_M$ be a point in $M$ where $u$ assumes its maximum, and let $\bar{x}$ be a point on $\partial M$ nearest to $x_M$ as geodesic distance. Let $\rho$ be the radius of the largest geodesic ball contained in $M$. Integrating (1), we have

$$\log \frac{\beta}{\beta - 1} \leq \left( \frac{n - 1}{1 - \alpha^2} \right)^{\frac{1}{2}} \rho \left[ \frac{(2n - 3)^2 + \alpha^2 (10n - 11)}{\alpha^2} (n - 1)K^2 + 2\{1 + 2(n - 1)\epsilon K\}^{\frac{1}{2}} R \\
+ \frac{2\beta}{\beta - 1} \left\{ 1 + 2(n - 1)\epsilon K \right\}^{\frac{1}{2}} \lambda_1 + C_1 \right]^{\frac{1}{2}}.$$ 

Hence we obtain the first Dirichlet eigenvalue

$$\lambda_1 \geq \frac{1}{\sqrt{1 + 2(n - 1)\epsilon K}} \frac{\beta - 1}{2\beta} \left\{ \frac{1 - \alpha^2}{n - 1} \frac{1}{\rho^2} (\log \frac{\beta}{\beta - 1})^2 - C \right\},$$

where

$$C = \frac{(2n - 3)^2 + \alpha^2 (10n - 11)}{\alpha^2} (n - 1)K^2 + 2\{1 + 2(n - 1)\epsilon K\}^{\frac{1}{2}} R + C_1.$$
Let
\[ f(\beta) = \frac{1}{2\sqrt{1 + 2(n-1)\varepsilon K}} \frac{\beta - 1}{\beta} \left\{ \frac{(1 - \alpha^2)}{(n-1)\rho^2} \left( \log \frac{\beta}{\beta - 1} \right)^2 - C \right\}, \]

Then \( f(\beta) \) has a maximum at \( \frac{\beta}{\beta - 1} = e^{1 + \sqrt{1 + \frac{(n-1)\rho^2 C}{\alpha^2}}} \). Hence

\[ \lambda_1 \geq \frac{1}{\sqrt{1 + 2(n-1)\varepsilon K}} \frac{(1 - \alpha^2)}{(n-1)\rho^2} (1 + B) \exp(-(1 + B)), \]

where

\[ B = \left\{ 1 + \frac{(n-1)\rho^2}{1-\alpha^2} C \right\}^{\frac{1}{2}}, \]

\[ C = \frac{(2n - 3)^2 + \alpha^2(10n - 11)}{\alpha^2} (n-1)K^2 + 2\{1 + 2(n-1)\varepsilon K\}^{\frac{1}{2}} R + C_1 \]

and

\[ C_1 = (n - 1)K \{ \frac{1}{\varepsilon} + 2(n - 1)(1 + 3H) \}. \]

References


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