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EXISTENCE THEOREM OF AN OPERATOR–VALUED SEQUENTIAL FUNCTION SPACE INTEGRAL

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In this paper we prove that an operator-valued sequential function space integral exists under some conditions and we investigate its properties. First of all, we present some necessary notations and definitions.

Let N, R and R_+ denote the set of all positive integers, the set of all real numbers and the set of all positive real numbers, respectively. Let C, C_+ and $C_+^\sim$ denote, respectively, the set of all complex numbers, the set of all complex numbers with positive real part and the set of all nonzero complex numbers with nonnegative real part. Let $C[a,b]$ denote the space of all real-valued continuous functions on $[a,b]$ and let $C_0[a,b]$ denote the subspace of $C[a,b]$ which vanishes at a; that is, the associated Wiener space. m_ω will denote Wiener measure on $C_0[a,b]$. In fact, every element y in $C[a,b]$ has a unique representation $y = x + \xi$ where x in $C_0[a,b]$ and ξ in R. Let $S[a,b]$ denote the space of all piecewise continuous functions on $[a,b]$. In this paper, we will be concerned with only the uniform topologies on $C[a,b]$, $C_0[a,b]$ and $S[a,b]$, and η will denote a Borel measure on $[a,b]$. Let $L^2(R)$ be the space of all C–valued Borel measurable functions on R which is square integrable with respect to Lebesgue measure m_L on R and let $\mathcal{L}(L^2(R))$ denote the space of all bounded linear operators from $L^2(R)$ into itself.

DEFINITION 1. For λ in $C_+^\sim$, the operator C_λ from $L^2(R)$ into itself is defined by

$$(C_\lambda\psi)(\xi) = \lambda^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \int_R \psi(x)\exp\Big\{-\frac{\lambda(x-\xi)^2}{2}\Big\}dm_L(x)$$

where ψ in $L^2(R)$ and $\lambda^{\frac{1}{2}}$ takes a nonnegative real part.

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REMARKS. (1) C_λ is a bounded linear operator from $L^2(R)$ into itself with $\|C_\lambda\| \le 1$ [5 ; p 776] and $\{C_\lambda | \lambda \in C_+ \bigcup \{0\}\}$ is a holomorphic semigroup where C_λ is the identity map if $\lambda = 0$ [8].

(2) For q in R_+, $C^{\#}_{-iq} = C_{iq}$ where $C^{\#}_{-iq}$ is the adjoint operator of C_{-iq} [5 ; p 776].

DEFINITION 2. The mixed norm space $L^*_{\infty 1:\eta}$ is defined by the space of all C–valued Borel measurable functions θ on $[a,b] \times R$ such that

$$\|\theta\|_{\infty 1:\eta} = \int_{[a,b]} \|\theta(s,\cdot)\|_\infty d|\eta|(s) < +\infty$$

and θ is $|\eta| \times m_L$ -almost everywhere (a.e.) continuous on $[a,b] \times R$ where $|\eta|$ is the total variation measure of η.

REMARKS. (1) $\|\theta(s,\cdot)\|_\infty$ is a Borel measurable function of s in $[a,b]$.

(2) Clearly $L^*_{\infty 1:\eta}$ equipped with the norm $\|\cdot\|_{\infty 1:\eta}$ becomes a normed linear space but not a Banach space.

DEFINITION 3. For s in $[a,b]$ and for θ in $L^*_{\infty 1:\eta}$, we define the multiplication operator $\theta(s)$ from $L^2(R)$ into itself by

$$(\theta(s)\psi)(\xi) = \theta(s,\xi)\psi(\xi) \qquad \text{for } \psi \text{ in } \ L^2(R).$$

REMARKS. (1) The operator $\theta(s)$ is well-defined for η-a.e. s in $[a,b]$ [4 ; p 8].

(2) $\theta(s)$ is a strongly measurable function of s in $[a,b]$ [8].

(3) $\theta(s)$ is a bounded linear operator with $\|\theta(s)\| = \|\theta(s,\cdot)\|_\infty$ for η-a.e.s in $[a,b]$ [6 ; p146, 4 ; p9].

Let $\sigma : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be any partition of $[a,b]$, and the norm of σ, denoted by $\|\sigma\|$, equals $\max\limits_{i=1,2,\cdots,n} (t_i - t_{i-1})$. For x in $C[a,b]$, we define

$$x_\sigma(t) \equiv \begin{cases} x(t_{i-1}) & \text{if } \ t_{i-1} \le t < t_i \quad (i = 1,2,\cdots,n) \\ x(b) & \text{if } \ t = b. \end{cases}$$

If σ is given as in the above and $\{v_0, v_1, \cdots, v_n\}$ is any set of $(n+1)$ real numbers, we define the function

$$z(\sigma; v_0, v_1, \cdots, v_n; t) \equiv \begin{cases} v_{i-1} & \text{if} \quad t_{i-1} \le t < t_i \quad (i = 1, 2, \cdots, n) \\ v_n & \text{if} \quad t = b. \end{cases}$$

Clearly, x_σ and $z(\sigma; v_0, v_1, \cdots, v_n; t)$ are in $S[a,b]$. And for a given functional F on $S[a,b]$, $F_\sigma(v_0, v_1, \cdots, v_n)$ is given by

$$f_\sigma\big(x(t_0), x(t_1), \cdots, x(t_n)\big) \equiv F\Big[z\big(\sigma; x(t_0), x(t_1), \cdots, x(t_n); \cdot\big)\Big]$$

for x in $C[a,b]$.

Let θ be in $L^*_{\infty 1:\eta}$ and let σ be given as in the above. Let

$$F(x) = \int_{[a,b]} \theta(s, x(s))\, d\eta(s) \quad \text{for } x \text{ in } S[a,b].$$

Then F is well-defined on $S[a,b]$. In fact,

$$f_\sigma\big(x(t_0), x(t_1), \cdots, x(t_n)\big) = \sum_{i=1}^{n} \int_{[t_{i-1}, t_i)} \theta\big(s, x(t_{i-1})\big)\, d\eta(s) + \theta\big(b, x(b)\big)\eta\big(\{b\}\big).$$

Hence for λ in R_+ and for each ψ in $L^2(R)$, by the Wiener integration formula and the change of variables $v_i = \lambda^{-\frac{1}{2}} u_i + v_0$ $(i = 1, 2, \cdots, n)$, we have

$$\begin{aligned}
&\lambda^{\frac{n}{2}} \big[(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})\big]^{-\frac{1}{2}} \int_{R^n} f_\sigma(v_0, v_1, \cdots, v_n) \psi(v_n) \cdot \\
&\qquad \cdot \exp\Big\{-\sum_{i=1}^{n} \frac{\lambda(v_i - v_{i-1})^2}{2(t_i - t_{i-1})}\Big\}\, d\mathop{X}_{i=1}^{n} m_L(v_i) \\
&= \big[(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})\big]^{-\frac{1}{2}} \int_{R^n} f_\sigma(v_0, \lambda^{-\frac{1}{2}} u_1 + v_0, \cdots, \lambda^{-\frac{1}{2}} u_n + v_0) \cdot \\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}} u_n + v_0) \exp\Big\{-\sum_{i=1}^{n} \frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})}\Big\}\, d\mathop{X}_{i=1}^{n} m_L(u_i) \\
&= \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}} x_\sigma + v_0)\, \psi(\lambda^{-\frac{1}{2}} x(b) + v_0)\, dm_\omega(x) \quad \text{where} \quad u_0 = 0.
\end{aligned}$$

Now we present the following definitions from the motivation specified above.

DEFINITION 4. Let λ be in C_+ and let ψ be in $L^2(R)$. Let F be a C-valued functional on $S[a,b]$. For a given partition $\sigma : a = t_0 < t_1 < \cdots < t_n = b$, the operator $I_\lambda^\sigma(F)$ is defined by the formula

$$(I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{\frac{n}{2}}\left[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\right]^{-\frac{1}{2}}\int_{R^n} f_\sigma(v_0,v_1,\cdots,v_n)\cdot$$
$$\cdot\,\psi(v_n)\exp\Big\{-\sum_{i=1}^{n}\frac{\lambda(v_i-v_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\mathop{X}_{i=1}^{n} m_L(v_i)$$

where $v_0 = \xi$, $f_\sigma(v_0,v_1,\cdots,v_n) = F\big[z(\sigma;v_0,v_1,\cdots,v_n;\cdot)\big]$ and $z(\sigma;v_0,v_1,\cdots,v_n;t)$ is defined as in the above. (If n is odd, we always choose $\lambda^{-\frac{1}{2}}$ with nonnegative real part). Let λ be in C_+ and let F be a functional on $S[a,b]$ such that $I_\lambda^\sigma(F)\psi$ exists for ψ in $L^2(R)$ and for every partition σ of $[a,b]$. If

$$\lim_{\|\sigma\|\to 0}\int_R (I_\lambda^\sigma(F)\psi)(\xi)\overline{\phi(\xi)}\,dm_L(\xi)$$

exists for all ψ,ϕ in $L^2(R)$, then this weak limit, $w-\lim\limits_{\|\sigma\|\to 0} I_\lambda^\sigma(F)$ is called the *operator-valued sequential function space integral* of F for λ in C^+ and denoted by $I_\lambda^{seq}(F)$; i.e.,

$$\big(I_\lambda^{seq}(F)\psi,\phi\big) = \lim_{\|\sigma\|\to 0}\big(I_\lambda^\sigma(F)\psi,\phi\big)$$

where the parenthesis $(\cdot,\cdot)$ denotes inner product.

DEFINITION 5. Let λ be in R_+ and let ψ be in $L^2(R)$. Let F be a C-valued functional on $C[a,b]$. Then the operator $I_\lambda(F)$ is defined by the Wiener integral

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x+\xi)\psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_\omega(x).$$

For λ in C_+, the operator-valued analytic function space integral of F, $I_\lambda^{an}(F)$ is defined to be the operator-valued function of λ which agree with $I_\lambda(F)$ for λ in R_+ and is analytic on C_+.

By the Fubini's Theorem and the Wiener integration formula,we have the following two lemmas [1]

LEMMA 6. *If $\theta(s,u)$ is $\eta \times m_L$-a.e. continuous on $[a,b] \times R$, then for almost all x in $C_0[a,b]$, $\theta(s,x(s))$ is continuous for η-a.e. s in $[a,b]$.*

LEMMA 7. *Let $\theta(s,u)$ be $\eta \times m_L$-a.e. continuous on $[a,b] \times R$ and let λ be in R_+. Then for $\eta \times m_L$-a.e. (x,ξ) in $C_0[a,b] \times R$, $\theta(s, \lambda^{-\frac{1}{2}}x(s)+\xi)$ is continuous for η-a.e. s in $[a,b]$.*

From [1; p 532], we have the following ;

LEMMA 8. *Let $\{x_n(z)\}$ be a sequence of vector-valued holomorphic functions in a domain D of the complex plane whose values lie in a separable Hilbert space H. Let $\|x_n(z)\| \le B$ for z in D. Let $\{z_k\}$ be a sequence of distinct points of D such that $\lim_{k\to\infty} z_k = z_0$ in D. Let $\lim_{n\to\infty} \big(x_n(z_k), y\big)$ exist for each y in H and each k. Then there exists $x(z)$ in H such that $\lim_{n\to\infty} \big(x_n(z), y\big) = \big(x(z), y\big)$ and $x(z)$ is holomorphic in D and $\lim_{n\to\infty} \big(x_n(z), y\big) = \big(x(z), y\big)$ uniformly on any compact subset of D for each y in H.*

THEOREM 9. *Let θ be in $L^*_{\infty 1:\eta}$ and let*

$$F(x) = \int_{[a,b]} \theta\big(s, x(s)\big)\, d\eta(s) \qquad \text{for } x \text{ in } S[a,b].$$

Then the operator-valued sequential function space integral of F^m, $I_\lambda^{seq}(F^m)$ exists for λ in C_+ and for m in N, and $\big(I_\lambda^{seq}(F^m)\psi, \phi\big) = \lim_{\|\sigma\|\to 0} \big(I_\lambda^{\sigma}(F^m)\psi, \phi\big)$ for all ψ, ϕ in $L^2(R)$. Moreover, $I_\lambda^{seq}(F^m) = I_\lambda^{an}(F^m)$.

Proof. Let $\sigma : a = t_0 < t_1 < \cdots < t_n = b$ be a given partition of $[a,b]$. Then F^m is well-defined on $S[a,b]$. In fact,

$$\begin{aligned}
&f_\sigma^m\big(x(t_0), x(t_1), \cdots, x(t_n)\big) \\
&= F^m\big[z(\sigma; x(t_0), x(t_1), \cdots, x(t_n); \cdot)\big] \\
&= \Big[\sum_{i=1}^{n} \int_{[t_{i-1}, t_i)} \theta\big(s, x(t_{i-1})\big)\, d\eta(s) + \theta\big(b, x(b)\big)\eta(\{b\})\Big]^m .
\end{aligned}$$

Thus we have formally that for λ in R_+ and for ψ in $L^2(R)$,

$$
\begin{aligned}
&\big(I_\lambda^\sigma(F^m)\psi\big)(\xi)\\
&= \lambda^{\frac{n}{2}}\big[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\big]^{-\frac{1}{2}}\int_{R^n} f_\sigma^m(v_0,v_1,\cdots,v_n)\cdot\\
&\qquad\cdot\psi(v_n)\exp\Big\{-\sum_{i=1}^n\frac{\lambda(v_i-v_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\overset{n}{\underset{i=1}{X}}m_L(v_i)\\
&=\big[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\big]^{-\frac{1}{2}}\int_{R^n} f_\sigma^m(v_0,\lambda^{-\frac{1}{2}}u_1+v_0,\cdots,\lambda^{-\frac{1}{2}}u_n+v_0)\cdot\\
&\qquad\cdot\psi(\lambda^{-\frac{1}{2}}u_n+v_0)\exp\Big\{-\sum_{i=1}^n\frac{(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\overset{n}{\underset{i=1}{X}}m_L(u_i)\\
&=\int_{C_0[a,b]} F^m(\lambda^{-\frac{1}{2}}x_\sigma+v_0)\,\psi(\lambda^{-\frac{1}{2}}x(b)+v_0)\,dm_\omega(x)
\end{aligned}
$$

where $u_0=0$ and $v_0=\xi$; that is,

$$
\big(I_\lambda^\sigma(F^m)\psi\big)(\xi)=\big(I_\lambda(F_\sigma^m)\psi\big)(\xi)\qquad\text{for }\lambda\text{ in }R_+.
$$

Next we will show that $\lim_{\|\sigma\|\to 0}\big(I_\lambda^\sigma(F^m)\psi,\phi\big)$ exists and equals $\big(I_\lambda(F^m)\psi,\phi\big)$ for λ in C_+. For λ in R_+ and for ψ,ϕ in $L^2(R)$,

$$
\begin{aligned}
&\lim_{\|\sigma\|\to 0}\big(I_\lambda^\sigma(F^m)\psi,\phi\big)\\
&=\lim_{\|\sigma\|\to 0}\int_R\big(I_\lambda^\sigma(F^m)\psi\big)(\xi)\overline{\phi(\xi)}\,dm_L(\xi)\\
&=\lim_{\|\sigma\|\to 0}\int_R\int_{C_0[a,b]} F^m(\lambda^{-\frac{1}{2}}x_\sigma+\xi)\,\psi(\lambda^{-\frac{1}{2}}x(b)+\xi)\,\overline{\phi(\xi)}\,dm_\omega(x)\,dm_L(\xi)
\end{aligned}
$$

$$\overset{(I)}{=} \int_R \lim_{\|\sigma\|\to 0} \int_{C_0[a,b]} F^m(\lambda^{-\frac{1}{2}} x_\sigma + \xi)\ \psi(\lambda^{-\frac{1}{2}} x(b) + \xi)\ \overline{\phi(\xi)}\, dm_\omega(x)\, dm_L(\xi)$$
$$\overset{(II)}{=} \int_R \int_{C_0[a,b]} \lim_{\|\sigma\|\to 0} F^m(\lambda^{-\frac{1}{2}} x_\sigma + \xi)\ \psi(\lambda^{-\frac{1}{2}} x(b) + \xi)\ \overline{\phi(\xi)}\, dm_\omega(x)\, dm_L(\xi)$$
$$\overset{(III)}{=} \int_R \int_{C_0[a,b]} F^m(\lambda^{-\frac{1}{2}} x + \xi)\ \psi(\lambda^{-\frac{1}{2}} x(b) + \xi)\ \overline{\phi(\xi)}\, dm_\omega(x)\, dm_L(\xi)$$
$$= \int_R (I_\lambda(F^m)\psi)(\xi)\overline{\phi(\xi)}\, dm_L(\xi)$$
$$= \big(I_\lambda(F^m)\psi, \phi\big).$$

Steps (I) and (II) result from the dominated convergence Theorem since $|F^m(\lambda^{-\frac{1}{2}} x_\sigma + \xi)| \le (\|\theta\|_{\infty 1:\eta})^m$ and $|\psi(\lambda^{-\frac{1}{2}} x(b) + \xi)|$ is Wiener integrable. Step (III) is obtained from the fact that

$$\lim_{\|\sigma\|\to 0} F^m(x_\sigma) = F^m(x)$$

since $\int_{[a,b]} \theta\big(s, x_\sigma(s)\big)\, d\eta(s) \le \int_{[a,b]} \|\theta(s,\cdot)\|_\infty\, d|\eta|(s) < +\infty$, and $\theta\big(s, x(s)\big)$ is continuous for η-a.e. s in $[a,b]$ from Lemma 6.

Using the Morera's Theorem, we can conclude that $\big(I_\lambda^\sigma(F^m)\psi, \phi\big)$ is holomorphic for λ in C_+ and so $I_\lambda^\sigma(F^m)\psi$ is holomorphic for λ in C_+ because the uniform analyticity is equivalent to the weak analyticity [8 ; p 189].

Consider the sequence $\{I_\lambda^\sigma(F^m)\psi\}$ of holomorphic functions. It is obvious that $\|I_\lambda^\sigma(F^m)\psi\| \le (\|\theta\|_{\infty 1:\eta})^m \|\psi\|$. Let $\{\lambda_n\}$ be a sequence of distinct points of R_+ such that $\lim_{n\to\infty} \lambda_n = \lambda$ in R_+. Then there exists $\lim_{\|\sigma\|\to 0} \big(I_{\lambda_n}^\sigma(F^m)\psi, \phi\big)$ for each ϕ in $L^2(R)$ and for each n in N. Therefore there exists the operator-valued sequential function space integral $I_\lambda^{seq}(F^m)$ such that $\big(I_\lambda^{seq}(F^m)\psi, \phi\big) = \lim_{\|\sigma\|\to 0} \big(I_\lambda^\sigma(F^m)\psi, \phi\big)$ and $I_\lambda^{seq}(F^m)\psi$ is holomorphic by Lemma 8. It follows from the uniqueness Theorem for a holomorphic function [3 ; p 97] that $I_\lambda^{seq}(F^m) = I_\lambda^{an}(F^m)$ for λ in C_+.

THEOREM 10. *Let $\sigma : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a given partition of $[a,b]$. Let θ be in $L^*_{\infty 1:\eta}$ and let F be defined as in Theorem*

9. If η is continuous on $[a,b]$; that is, $\eta(\{\tau\})=0$ for each τ in $[a,b]$, then for λ in C_+ and for ψ in $L^2(R)$, $I_\lambda^\sigma(F^m)\psi$ is in $\mathcal{L}(L^2(R))$ and

$$\Big(I_\lambda^\sigma(F^m)\psi\Big)(\xi)=m!\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq 0}}\int_{\prod_{i=1}^{n}\Delta_{[t_{i-1},t_i),k_i}}\Big(\mathcal{L}_\sigma\circ\psi\Big)(\xi)\,d\mathop{X}_{i=1}^{n}\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})$$

where $\Delta_{[t_{i-1},t_i),k_i}=\big\{(s_{i-1,1},s_{i-1,2},\cdots,s_{i-1,k_i})\in[t_{i-1},t_i)^{k_i}\,|\,t_{i-1}<s_{i-1,1}<s_{i-1,2}<\cdots<s_{i-1,k_i}<t_i\big\}$ for $i=1,2,\cdots,n$ and the composition operator,
$\mathcal{L}_\sigma=\theta(s_{0,1})\circ\theta(s_{0,2})\circ\cdots\circ\theta(s_{0,k_1})\circ C_{\frac{\lambda}{t_1-t_0}}\circ\theta(s_{1,1})\circ\theta(s_{1,2})\circ\cdots\circ\theta(s_{1,k_2})\circ C_{\frac{\lambda}{t_2-t_1}}\circ\cdots\circ C_{\frac{\lambda}{t_{n-1}-t_{n-2}}}\circ\theta(s_{n-1,1})\circ\theta(s_{n-1,2})\circ\cdots\circ\theta(s_{n-1,k_n})\circ C_{\frac{\lambda}{t_n-t_{n-1}}}$ for $(s_{i-1,1},s_{i-1,2},\cdots,s_{i-1,k_i})$ in $\Delta_{[t_{i-1},t_i),k_i}$. Moreover

$$\|I_\lambda^\sigma(F^m)\|\leq\big(\|\theta\|_{\infty 1:\eta}\big)^m\quad\text{for }\lambda\text{ in }C_+.$$

Proof. Let λ be in C_+ and let ψ be in $L^2(R)$. Then

$$\begin{aligned}
&(I_\lambda^\sigma(F^m)\psi)(\xi)\\
&\overset{(1)}{=}\lambda^{\frac{n}{2}}\left[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\right]^{-\frac12}\int_{R^n}\Big[\sum_{i=1}^{n}\int_{[t_{i-1},t_i)}\theta(s,u_{i-1})\,d\eta(s)\Big]^m\cdot\\
&\cdot\psi(u_n)\exp\Big\{-\sum_{i=1}^{n}\frac{\lambda(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\mathop{X}_{i=1}^{n}m_L(u_i)\\
&\overset{(2)}{=}\lambda^{\frac{n}{2}}\left[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\right]^{-\frac12}\int_{R^n}\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq0}}\frac{m!}{k_1!k_2!\cdots k_n!}\cdot\\
&\cdot\prod_{i=1}^{n}\Big(\int_{[t_{i-1},t_i)}\theta(s,u_{i-1})\,d\eta(s)\Big)^{k_i}\psi(u_n)\exp\Big\{-\sum_{i=1}^{n}\frac{\lambda(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\mathop{X}_{i=1}^{n}m_L(u_i)
\end{aligned}$$

$$\overset{(3)}{=}\lambda^{\frac{n}{2}}\left[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\right]^{-\frac{1}{2}}\int_{R^n}\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq 0}} m!\cdot$$

$$\cdot\int_{\prod_{i=1}^{n}\Delta_{[t_{i-1},t_i),k_i}}\prod_{i=1}^{n}\prod_{v=1}^{k_i}\theta(s_{i-1,v},u_{i-1})\,d\mathop{X}_{i=1}^{n}\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})\psi(u_n)\cdot$$

$$\cdot\exp\Big\{-\sum_{i=1}^{n}\frac{\lambda(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\mathop{X}_{i=1}^{n}m_L(u_i)$$

$$\overset{(4)}{=}m!\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq 0}}\int_{\prod_{i=1}^{n}\Delta_{[t_{i-1},t_i),k_i}}\lambda^{\frac{n}{2}}\left[(2\pi)^n(t_1-a)\cdots(t_n-t_{n-1})\right]^{-\frac{1}{2}}\cdot$$

$$\cdot\int_{R^n}\prod_{i=1}^{n}\prod_{v=1}^{k_i}\theta(s_{i-1,v},u_{i-1})\psi(u_n)\exp\Big\{-\sum_{i=1}^{n}\frac{\lambda(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\Big\}\,d\mathop{X}_{i=1}^{n}m_L(u_i)\cdot$$

$$\cdot\,d\mathop{X}_{i=1}^{n}\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})$$

$$\overset{(5)}{=}m!\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq 0}}\int_{\prod_{i=1}^{n}\Delta_{[t_{i-1},t_i),k_i}}(\mathcal{L}_\sigma\circ\psi)(\xi)\,d\mathop{X}_{i=1}^{n}\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})$$

where $u_0=\xi$. Step (1) result from the continuity of η and the definition of $I_\lambda^\sigma(F^m)$. By the multinomial expansion, we have Step (2). Let $A_{\alpha,\beta}=\{(s_{i-1,1},s_{i-1,2},\cdots,s_{i-1,k_i})\in[t_{i-1},t_i)^{k_i}\,|\,s_{i-1,\alpha}=s_{i-1,\beta}\}$. Then by the Fubini's Theorem, $A_{\alpha,\beta}$ is a $\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})$- null set. Let P_{k_i} be the set of all permutations of $\{1,2,\cdots,k_i\}$ and let for τ in P_{k_i}, $\Delta_{\tau(k_i)}=\{(s_{i-1,1},s_{i-1,2},\cdots,s_{i-1,k_i})\in[t_{i-1},t_i)^{k_i}\,|\,t_{i-1}<s_{i-1,\tau(1)}<s_{i-1,\tau(2)}<\cdots<s_{i-1,\tau(k_i)}<t_i\}$. Then $[t_{i-1},t_i)^{k_i}=[\bigcup_{\tau\in P_{k_i}}\Delta_{\tau(k_i)}]\bigcup[\bigcup_{1\leq\alpha\neq\beta\leq k_i}A_{\alpha,\beta}]$. Since the integrand is invariant under permutations of s-variables, the integrals over the $k_i!$ simplexes are equal [4]. Hence Step (3) follows. From the Fubini's Theorem, we obtain Step (4) which will be justified below in conjunction with the proof of the norm estimate. Using Definitions 1 and 3, Step (5) is valid.

Moreover

$$
\begin{aligned}
&\|I_\lambda^\sigma(F^m)\psi\| \\
&= \Big\| m! \sum_{\substack{k_1+k_2+\cdots+k_n=m \\ k_i\geq 0}} \int_{\prod_{i=1}^n \Delta_{[t_{i-1},t_i),k_i}} (\mathcal{L}_\sigma \circ \psi) \, d \mathop{X}_{i=1}^{n} \mathop{X}_{v=1}^{k_i} \eta(s_{i-1,v}) \Big\| \\
&\overset{(I)}{\leq} m! \sum_{\substack{k_1+k_2+\cdots+k_n=m \\ k_i\geq 0}} \int_{\prod_{i=1}^n \Delta_{[t_{i-1},t_i),k_i}} \prod_{i=1}^{n}\prod_{v=1}^{k_i} \|\theta(s_{i-1,v})\| \|\psi\| \, d \mathop{X}_{i=1}^{n} \mathop{X}_{v=1}^{k_i} |\eta|(s_{i-1,v}) \\
&\overset{(II)}{=} \sum_{\substack{k_1+k_2+\cdots+k_n=m \\ k_i\geq 0}} \frac{m!}{k_1!k_2!\cdots k_n!} k_1!k_2!\cdots k_n! \int_{\prod_{i=1}^n \Delta_{[t_{i-1},t_i),k_i}} \prod_{i=1}^{n}\prod_{v=1}^{k_i} \cdot \\
&\qquad \cdot \|\theta(s_{i-1,v})\| \|\psi\| \, d \mathop{X}_{i=1}^{n} \mathop{X}_{v=1}^{k_i} |\eta|(s_{i-1,v}) \\
&\overset{(III)}{=} \sum_{\substack{k_1+k_2+\cdots+k_n=m \\ k_i\geq 0}} \frac{m!}{k_1!k_2!\cdots k_n!} \prod_{i=1}^{n} \Big(\int_{[t_{i-1},t_i)} \|\theta(s)\| \, d|\eta|(s) \Big)^{k_i} \|\psi\| \\
&\overset{(IV)}{=} \Big(\int_{[a,b)} \|\theta(s)\| \, d|\eta|(s) \Big)^m \|\psi\| \\
&\overset{(V)}{\leq} \left(\|\theta\|_{\infty 1:\eta}\right)^m \|\psi\| \\
&< +\infty
\end{aligned}
$$

Step (I) follows from the Remark (1) of Definition 1 and the generalized Minkowski's inequality. Steps (II),(III) and (IV) follow from the reversed process of the calculation above. From the Remark (3) of Definition 3, we have Step (V). By the result above, we have $I_\lambda^\sigma(F^m)\psi$ is in $L^2(R)$; i.e., $I_\lambda^\sigma(F^m)\psi$ is in $\mathcal{L}\big(L^2(R)\big)$.

Finally, letting $F(x) = \int_{[a,b]} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s)$, we have $\big(I_\lambda^\sigma(F)|\psi|\big)(\xi) \leq \|\theta\|_{\infty 1:\eta} C_{\frac{\lambda}{b-a}} |\psi|(\xi)$. This justifies the use of the Fubini's Theorem in Step (4) above.

In [4 ; p 22], it is shown that for $F^m(x) = \Big(\int_{(a,b)} \theta\big(s, x(s)\big) \, d\eta(s)\Big)^m$

on x in $C[a,b]$, $I_\lambda^{an}(F^m)$ exists for λ in $C_+^\sim$ and

$$I_\lambda^{an}(F^m) = m! \int_{\Delta_m} \mathcal{L}\, d \mathop{X}_{i=1}^{m} \eta(s_i)$$

where $\Delta_m = \{(s_1, s_2, \cdots, s_m) \in (a,b)^m \mid a < s_1 < s_2 < \cdots < s_m < b\}$ and

$$\mathcal{L} = C_{\frac{\lambda}{s_1 - a}} \circ \theta(s_1) \circ C_{\frac{\lambda}{s_2 - s_1}} \circ \theta(s_2) \circ \cdots \circ C_{\frac{\lambda}{s_m - s_{m-1}}} \circ \theta(s_m) \circ C_{\frac{\lambda}{b - s_m}}$$

for $(s_1, s_2, \cdots, s_m)$ in Δ_m.

From the result above and Theorems 9 and 10, we have the following Theorem ;

THEOREM 11. *Under the same assumptions and notations as in the above,*

$$w - \lim_{\|\sigma\| \to 0} \sum_{\substack{k_1+k_2+\cdots+k_n=m \\ k_i \geq 0}} \int_{\prod_{i=1}^{n} \Delta_{[t_{i-1}, t_i), k_i}} \mathcal{L}_\sigma\, d \mathop{X}_{i=1}^{n} \mathop{X}_{v=1}^{k_i} \eta(s_{i-1,v}) = \int_{\Delta_m} \mathcal{L}\, d \mathop{X}_{i=1}^{m} \eta(s_i).$$

DEFINITION 12. For θ in $L^*_{\infty 1:\eta}$, let $\mathcal{M}_\theta$ be the set of all functionals G on $S[a,b]$ of the form

$$G(x) = \sum_{m=0}^{\infty} a_m \Big(\int_{[a,b]} \theta(s, x(s))\, d\eta(s) \Big)^m$$

such that $g(z) = \sum_{m=0}^{\infty} a_m z^m$ is an analytic function with the radius of convergence greater than $\|\theta\|_{\infty 1:\eta}$. For G in $\mathcal{M}_\theta$, we define the norm of G

$$\|G\| = \sum_{m=0}^{\infty} |a_m| \big(\|\theta\|_{\infty 1:\eta} \big)^m.$$

REMARK. We can easily check that $(\mathcal{M}_\theta, \|\cdot\|)$ is a normed linear space for each θ in $L^*_{\infty 1:\eta}$.

THEOREM 13. *Let G be in $\mathcal{M}_\theta$ such that*

$$G(x) = \sum_{m=0}^{\infty} a_m \Big(\int_{[a,b]} \theta(s, x(s))\, d\eta(s) \Big)^m .$$

Then for λ in C_+, $I_\lambda^{seq}(G)$ exists in $\mathcal{L}(L^2(R))$ and equals $\sum_{m=0}^{\infty} a_m I_\lambda^{seq}(F^m)$ in the uniform operator topology where $F(x)$ is given as in Theorem 9.

Proof. Let λ be in C_+ and let $\sigma : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a given partition of $[a, b]$. Then

$$\begin{aligned}
&(I_\lambda^\sigma(G)\psi)(\xi) \\
&\overset{(1)}{=} \lambda^{\frac{n}{2}} \big[(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})\big]^{-\frac{1}{2}} \int_{R^n} \sum_{m=0}^{\infty} a_m \Big\{ \sum_{i=1}^{n} \int_{[t_{i-1}, t_i)} \theta(s, u_{i-1})\, d\eta(s) \\
&\qquad + \theta(b, u_n)\eta(\{b\}) \Big\}^m \psi(u_n) \exp\Big\{ -\sum_{i=1}^{n} \frac{\lambda(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \Big\}\, d \mathop{X}_{i=1}^{n} m_L(u_i) \\
&\overset{(2)}{=} \sum_{m=0}^{\infty} a_m \lambda^{\frac{n}{2}} \big[(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})\big]^{-\frac{1}{2}} \int_{R^n} \Big\{ \sum_{i=1}^{n} \int_{[t_{i-1}, t_i)} \theta(s, u_{i-1}) \cdot \\
&\qquad \cdot d\eta(s) + \theta(b, u_n)\eta(\{b\}) \Big\}^m \psi(u_n) \exp\Big\{ -\sum_{i=1}^{n} \frac{\lambda(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \Big\}\, d \mathop{X}_{i=1}^{n} m_L(u_i) \\
&\overset{(3)}{=} \sum_{m=0}^{\infty} a_m (I_\lambda^\sigma(F^m)\psi)(\xi)
\end{aligned}$$

Steps (1) and (3) result from the definition of the operator-valued sequential function space integral. Since the integrand in the right-hand side of the equality (1) is dominated by

$$\sum_{m=0}^{\infty} |a_m| (\|\theta\|_{\infty 1:\eta})^m \exp\{ -\sum_{i=1}^{n} \frac{(Re\lambda)(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \} |\psi(u_n)|,$$

we obtain Step (2) from the dominated convergence Theorem. Let ψ and ϕ be in $L^2(R)$. Because $\sum_{m=0}^{\infty} |a_m| (\|\theta\|_{\infty 1:\eta})^m \|\psi\| \|\phi\|$ is a convergent series, the double sequence $\left\{\sum_{m=0}^{\infty} a_m\big(I_\lambda^\sigma(F^m)\psi,\phi\big)\right\}$ converges uniformly for k in N [2 ; p 28]; that is,

$$\lim_{k\to\infty}\lim_{\|\sigma\|\to 0}\sum_{m=0}^{k} a_m\big(I_\lambda^\sigma(F^m)\psi,\phi\big)=\lim_{\|\sigma\|\to 0}\sum_{m=0}^{\infty} a_m\big(I_\lambda^\sigma(F^m)\psi,\phi\big).$$

By the previous results, we have

$$\begin{aligned}&\Big(\sum_{m=0}^{\infty} a_m I_\lambda^{seq}(F^m)\psi,\phi\Big)\\ &=\lim_{k\to\infty}\Big(\sum_{m=0}^{k} a_m \lim_{\|\sigma\|\to 0} I_\lambda^\sigma(F^m)\psi,\phi\Big)\\ &=\lim_{\|\sigma\|\to 0}\Big(\sum_{m=0}^{\infty} a_m I_\lambda^\sigma(F^m)\psi,\phi\Big)\\ &=\lim_{\|\sigma\|\to 0}\big(I_\lambda^\sigma(G)\psi,\phi\big)\qquad \text{for } \psi,\phi \text{ in } \quad L^2(R).\end{aligned}$$

Hence $I_\lambda^{seq}(G)=w-\lim\limits_{\|\sigma\|\to 0} I_\lambda^\sigma(G)$ exists and we obtain

$$I_\lambda^{seq}(G)=\sum_{m=0}^{\infty} a_m I_\lambda^{seq}(F^m).$$

We have the following corollary directly from Theorems 10 and 13 by letting $a_m=\dfrac{1}{m!}$ in the infinite series representation of G in $\mathcal{M}_\theta$.

COROLLARY 14. *For θ in $L^*_{\infty 1:\eta}$, let*

$$H(x)=\exp\Big(\int_{[a,b]}\theta\big(s,x(s)\big)\,d\eta(s)\Big)\quad on\quad S[a,b].$$

Then $I_\lambda^{seq}(H)$ exists for λ in C_+. If η is continuous on $[a,b]$, then

$$I_\lambda^{seq}(H)=w-\lim_{\|\sigma\|\to 0}\sum_{m=0}^{\infty}\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_i\geq 0}}\int_{\prod_{i=1}^{n}\Delta_{[t_{i-1},t_i),k_i}}\mathcal{L}_\sigma\, d\mathop{X}_{i=1}^{n}\mathop{X}_{v=1}^{k_i}\eta(s_{i-1,v})$$

where $\mathcal{L}_\sigma$ is as in Theorem 10.

For θ in $L^*_{\infty 1:\eta}$ and for λ in C_+, consider a mapping $I_{\lambda,\theta} : \mathcal{M}_\theta \longrightarrow \mathcal{L}\big(L^2(R)\big)$ with $I_{\lambda,\theta}(G) = I_\lambda^{seq}(G)$ for G in $\mathcal{M}_\theta$. Then by the elementary properties of integration, $I_{\lambda,\theta}$ is linear.

THEOREM 15. *For λ in C_+ and for θ in $L^*_{\infty 1:\eta}$, $I_{\lambda,\theta}$ is a bounded linear operator with the norm $|||I_{\lambda,\theta}||| \leq 1$.*

Proof. Since $\|I_{\lambda,\theta}(G)\| \leq \sum_{m=0}^{\infty} |a_m|\big(\|\theta\|_{\infty 1:\eta}\big)^m$ for

$$G(x) = \sum_{m=0}^{\infty} a_m \Big(\int_{[a,b]} \theta\big(s, x(s)\big)\, d\eta(s)\Big)^m$$

in $\mathcal{M}_\theta$, from the definition of $\|G\|$, we have

$$|||I_{\lambda,\theta}||| = \sup_{\|G\|\neq 0} \frac{\|I_{\lambda,\theta}(G)\|}{\|G\|} \leq 1.$$

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