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AUXILIARY PRINCEPLE AND ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES

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Abstract

The auxiliary principle technique is used to prove the uniqueness and the existence of solutions for a class of nonlinear variational inequalities and suggest an innovative iterative algorithm for computing the approximate solution of variational inequalities. Error estimates for the finite element approximation of the solution of variational inequalities are derived, which refine the previous known results. An example is given to illustrate the applications of the results obtained. Several special cases are considered and studied.

1. Introduction

Variational inequality theory has become an effective technique for studying a wide class of problems arising in various branches of mathematical and engineering sciences in a natrual, unified and general framework. This theory has been generalized and extended in several directions using new and powerful methods that have led to the solutions of basic and fundamental problems thought to be in accessible proviously. Some of these developments have made mutually enriching contacts with other areas of pure and applied sciences. Variational inequality theory as developed by the Italian and French schools in the early 1960s and thereafter, constituted a significant extension of the variational principles. It has been shown recently that the developement of the variational inequality theory led to a number of advances in the study of contact problems in solid mechanics, the general theory of transportation and economics equilibrium and fluid flow through porous media etc. The variety of problems

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to which variational inequality techniques may be applied is impressive and amply representative for the richness of the field. One of the charms of this theory is that the location of the free boundary (contact area) becomes an instrinsic part of the solution and no special devices are needed to locate it.

The development of the variational inequality theory can be viewed as the simultaneous pursuits of two different lines of research during the last two decades: On the one hands, it reveals the fundamental facts on the qualitative behaviour of solutions (regarding existence, uniqueness and regularity) to important classes of nonlinear boundary value problems; on the other hand, it also provides a mean for developing highly efficient new approximation and numerical methods to solve, for example, free and moving boundary problems and complementarity problems. In most cases, the issue of the existence of solutions to variational inequalities related to the contact problems is an open question. Special cases have been considered by Kikuchi and Oden [1], Noor [2, 3, 4], Glowinski, Lions and Tremolieres [5] and Duvaut and Lions [6].

There is already in the literature a substantial number of iterative type algorithms including the projection method for finding the numerical solution of variational inequalities, see [1, 5, 6] and the reference therein. It is worth mentioning that the scope of the projection algorithm is quite limited due to the fact that it is very difficult to find the projection of the space into the convex set except in very simple cases. Secondly, the projection cannot be applied for other classes of variational inequalities of type (3.1). These facts motivated Noor [2, 4] and Glowinski, Lions and Tremolieres [5] to develop alternative methods to study the existence of solution of variational inequalities. This approach deals with the auxiliary variational inequalities. This approach deals with the auxiliary variational inequality problem and proving that the solution of the auxiliary problem is the solution of the variational inequality problem. It turns out that this technique is equivalent to finding the minimum of the functional associated with the auxiliary variational inequality problem on the convex set in the space. This technique provides us with a general framework to suggest and analyze an innovative and novel iterative algorithm for computing the solution of variational inequalities. For related work, see Cohen [7], where he has shown that the auxiliary principle technique provides us a general framework to describe and analyze many computational algorithms ranging from gradient to decomposition / coordination algorithms.

Inspired and motivated by the research work going on in this field and related areas, we consider a class of nonlinear variational inequalities. Using the auxiliary principle technique of Noor [4] and Glowinski, Lions and Tremolieres [5], we prove the existence of the unique solution of this class and suggest a novel and general algorithm. Abstract error estimates are obtained by using the finite element approximate solution of the class of variational inequalities. An example is given to illustrate the techniques and applications of the theory developed in this paper.

2. Preliminaries and formulations

Let H be a real Hilbert space with its dual space H', whose inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. The pairing between elements of H' and H is denoted by $\langle\cdot,\cdot\rangle$. Let M be a closed convex nonempty subset of H.

Let $T : H \to H'$ be a continuous operator and f be a real-valued continuous functional on H. We consider the functional,

$$I[v] = \langle Tv, v\rangle - 2f(v) \tag{2.1}$$

which is known as the energy (cost) functional. It is worth mentioning that a wide class of linear and nonlinear problems arising in mathematical and engineering sciences either arise or can be formulated in terms of functional of this form (2.1). Here one usually seeks to minimizes the functional $I[v]$, defined by (2.1) over a whole space or on a convex set in H, keeping in mind whether the real-valued functional f is linear or nonlinear differentiable. We point out that the whole theory of variational methods can be based on the minimum of the functional $I[v]$.

We now consider the following cases, in which T is a linear and symmetric operator.

I. For given $f \in H'$, it is known [1] that the minimum of $I[v]$ on M in H is equivalent to finding $u \in M$ such that

$$\langle Tu, v-u\rangle \geq \langle f, v-u\rangle, \text{ for all } v \in M. \tag{2.2}$$

The inequality (2.2) is known as variational inequality.

II. For a differentiable nonlinear function f, Noor [8] has proved that the minimum of $I[v]$ on M can be characterized by a class of variational inequalities of the form

$$\langle Tu, v-u\rangle \geq \langle f', v-u\rangle, \text{ for all } v \in M, \tag{2.3}$$

where $f'(u)$ is the Frecher differential of f at u. The inequality of type (2.3) is called the mildly nonlinear variational inequality. For the existence of the solution and error estimates; see Noor [8, 9]. Recently, it has been shown by Panagiotopoulas [10] that unilateral problems with non-convex potential can only be characterized by the inequality of type (2.3).

III. For a non-differentiable nonlinear functional f, the minimum of $I[v]$ on M can be characterized by a class of variational inequalities of the type:

$$(2.4) \qquad \langle Tu, v-u\rangle + f_1(v) - f_1(u) \geq \langle f_2'(u), v-u\rangle, \quad \text{for all } v \in M,$$

where $f(u) = f_1(u)+f_2(u)$, with f_1 non-differentiable and f_2 differentiable functionals respectively, see Noor [11].

Clearly variational inequalities of type (2.4) are more general and includes (2.2), and (2.3) as special cases. It is also obvious from the above facts that the nonlinear programming problems and variational inequalities are equivalent. This interplay between variational inequalities and nonlinear programming problems is very subtle and has been fruitful. This equivalence has been used to suggest many unified and general algorithms for various classes of complementarity problems, see Noor [12, 13, 14] and Ahn [15]. In fact, variational inequalities are more general than and include many mathematical programming problems as special case. However, in many important applications problems like (2.4) occur which involve non- symmetric nonlinear operators T in mathematical models for problems in engineering and physical sciences. This fact alone motivates the interest of studying problem (2.4) on its own, that is without assuming " a priori " that it comes out as an Euler inequality of an extremum problem.

3. Existence Results

In this section, we study those conditions under which there does exist a unique solution of more general variational inequality of which (2.4) is a special case.

Let us consider the following problem:

PROBLEM 3.1. *Let T, A be nonlinear operators such that $A(u) \in H'$. If the functional $j : H \to R$ is convex, lower semi-continuous and proper, then find $u \in M$ such that*

$$(3.1) \qquad \langle Tu, v-v\rangle + j(v) - j(u) \geq \langle A(u), v-u\rangle, \quad \textit{for all } v \in M.$$

It is obvious that variational inequalities of type (2.2) - (2.4) are special cases of (3.1).

We also define the following concepts.

DEFINITION 3.1. *The nonlinear operator $T : M \to H'$ is called:*
(1) Strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \quad \text{for all } u, v \in M.$$

(2) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \quad \text{for all } u, v \in M.$$

(3) Antimonotone, if for all $u, v \in M$,

$$\langle Tu - Tv, u - v\rangle \leq 0.$$

It is obvious that strongly monotonicity implies monotonicity, but not conversely. In particular, $\alpha \leq \beta$.

Finally, we define Λ, a canonical isomorphism from H' onto H, for all $f \in H'$, such that

$$\langle f, u\rangle = (\Lambda f, u), \quad \text{for all } v \in M. \tag{3.2}$$

Then $\|\Lambda\|_{H'} = 1 = \|\Lambda^{-1}\|_H$.

We make the following hypothesis.

CONDITION N. *We assume that $\gamma < \alpha$, where α is the strongly monotonicity constant of T snf γ is the Lipschitz constant of the operator A.*

We now state and prove the main result of this section.

THEOREM 3.1. *Let T be a strongly monotone and Lipschitz continuous operator. If the nonlinear operator A is Lipschitz continuous antimonotone, and condition N holds, then there exists a unique solution $u \in M$ satisfying (3.1), where $j(\cdot)$ is convex, lower semi-continuous and proper functional.*

Proof.

(a) Uniqueness: Its proof is similar to that of Noor [2] and Glowinski, Lions and Tremolieres [5].

(b) Existence: We now use the auxiliary principle technique of Noor [2, 4] to prove the existence of a solution of (3.1).

For each $u \in M$ and $\rho > 0$, we consider the auxiliary problem of finding $w \in M$ satisfying the variational inequality.

(3.3) $(w, v-w)+\rho j(v)-\rho j(w) \geq (u, v-w)+\rho\langle A(u), v-w\rangle-\rho\langle Tu, v-w\rangle,$

Let w_1, w_2 be two solutions of (3.3) related to $u_1, u_2 \in M$ respectively. It is enough to show that the mapping $u \to w$ has a fixed point belonging to the closed convex set M in H satisfying (3.1). In other words, we have to show that for all ρ well chosen,

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2.

Taking $v = w_2$ (respectively w_1) in (3.3) related to u_1 (respectively u_2), we obtain

$$\begin{aligned}(w_1, w_2 - w_1) + \rho j(w_2) - \rho j(w_1) &\geq (u_1, w_2 - w_1)\\ &+\rho\langle A(u_1), w_2 - w_1\rangle - \rho\langle Tu_1, w_2 - w_1\rangle.\\ (w_2, w_1 - w_2) + \rho j(w_1) - \rho j(w_2) &\geq (u_2, w_1 - w_2)\\ &+\rho\langle A(u_1), w_2 - w_1\rangle - \rho\langle Tu_2, w_2 - w_1\rangle.\end{aligned}$$

Adding these inequalities and using (3.2), we have

$$\begin{aligned}(w_1 - w_2, w_1 - w_2) \leq &(u_1 - u_2 - \rho\Lambda(Tu_1 - Tu_2), w_1 - w_2)\\ &+ \rho(\Lambda(A(u_1) - A(u_2)), w_1 - w_2)\end{aligned}$$

Now using strongly monotonicity of T and Lipschitz continuity of T and A, we have, (see Noor [16] for details),

$$\begin{aligned}\|w_1 - w_2\| &\leq \|u_1 - u_2 - \rho\Lambda(Tu_1 - Tu_2)\| + \rho\|A(u_1) - A(u_2)\|\\ &\leq ((\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}) + \rho\gamma))\|u_1 - u_2\|\\ &= \theta\|u_1 - u_2\|,\end{aligned}$$

where $\theta = (\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}) + \rho\gamma < 1$ for $\rho\gamma < 1$ and $0 < \rho < \frac{2(\alpha-\gamma)}{\beta^2-\gamma^2}$, by condition N.

This shows that the mapping $u \to w$ defined by (3.3) has a fixed point, which is the solution of (3.1), the required result.

REMARK 3.1. It is clear that for each $u \in K$ and $\rho > 0$, $w \in K$ satisfying (3.3) is equivalent to finding the minimum of the functional $I_1[v]$ on K,

$$I_1[v] = \frac{1}{2}\langle v, v\rangle + \langle \rho(Tu - A(u)) - u, v\rangle + \rho j(v),$$

which is the convex differentiable functional associated with the auxiliary variational inequality (3.3).

Following the ideas of Noor [2,4] and Cohen [7], we propose and analyze a general iterative algorithm.

For some $u \in K$, we introduce the following general auxiliary problem of finding the minimum of the functional $F[w]$ on K in H, where

$$F[w] = E(w) + \langle \rho(Tu - A(u)) - E'(u), w\rangle + \rho j(w). \tag{3.4}$$

Here $E(w)$ is a convex differentiable functional and $\rho > 0$ is a constant. It is clear that the minimum of $F[w]$, defined by (3.4) can be characterized by a variational inequality

$$\langle E'(w), v-w\rangle + \rho j(v) - j(w) \geq \langle E'(u), v-w\rangle + \rho\langle A(u), v-w\rangle - \rho\langle Tu, v-w\rangle, \tag{3.5}$$

for all $v \in K$.

It is obvious that the auxiliary variational inequality (3.3) is a special case of (3.5). We also note that if $w = u$, then w is a solution of the variational inequality problem (3.1).

We note that in may applications, the auxiliary variational inequalities (3.5) occur, which do not arise as a result of minimization problems. This motivates the interest of studying problems (3.3) and (3.5) on its own, that is without assuming 'a priori ' that these come out as an Euler inequality of an extremum problem. The main motivation of this section is to suggest a general auxiliary variational inequality problem, which includes (3.3) and (3.5) as special cases.

AUXILIARY PRINCIPLE 3.1. *For some $w \in K$, we consider the general auxiliary variational inequality problem of finding $w \in K$ such that*

$$\langle M(w), v - w\rangle + \rho j(v) - \rho j(v) - \rho j(w)$$

$$\geq \langle M(u), v - w\rangle + \rho\langle A(u), v - w\rangle - \rho\langle Tu, v - w\rangle, \tag{3.6}$$

for all $v \in K$, where $\rho > 0$ is a constant and M is a nonlinear continuous operator.

It is clear that the problems (3.3) and (3.5) can be derived from the auxiliary problem (3.6). Obviously, if $w = u$, then w is a solution of the nonlinear variational inequality problem (3.1).

Based on these observations, we now suggest and analyze the following algorithm.

ALGORITHM 3.1.

(a) At $n = 0$, start with some initial w_0.

(b) At step n, solve the auxiliary problem (3.6) with $u = w_n$. Let w_{n+1} denote the solution of the problem (3.6).

(c) If $\|w_{n+1} - w_n\| \leq \varepsilon$, for given $\varepsilon > 0$ stop. Otherwise, repeat (b).

SPECIAL CASE 3.1.

I. If $j(v) \equiv 0$, then we have new proof for the existence of a solution of a variational inequality

$$\langle Tu, v - u\rangle \geq \langle A(u), v - u\rangle, \qquad \text{for all } v \in M.$$

For other proofs, see Noor [9, 11]

II. If $A(u) \equiv f$ then our result is the same as proved in [1], that is, find $u \in M$ such that

$$\langle Tu, v - u\rangle + j(v) - j(u) \geq \langle f, v - u\rangle, \qquad \text{for all } v \in M.$$

III. If $A(u) \equiv f \in H$, and $j(v) \equiv 0$, then problem (3.1) becomes:
For given $f \in H'$, find $u \in M$ such that

$$\langle Tu, v - u\rangle \geq \langle f, v - u\rangle, \qquad \text{for all } v \in M.$$

a problem originally considered and studied by Browder [17] and Hartman and Stampacchia [18].

4. Abstract Error Estimates

We now derive a general error estimate for the finite element approximation of the solution of variational inequalities of type (3.1). Our estimates are quite general. They hold for any finite dimensional subspace S_h and approximate constraint set M_h and represent a significant improvement of all the estimates for corresponding elliptic variational inequalities. For definiteness, we shall assume that there exists a Hilbert space U which is densely and continuously embedded in the dual space H'. It is then possible to identify H with α subspace of U' that is dense in U' by a continuous injection.

In order to derive the error estimate for the approximate solutions for variational inequalities of type (3.1). Let $S_h \subset H$ be a finite dimensional subspace and $M_h \subset H$ be a finite dimensional convex set. An approximation of (3.1) is that of finding $u_h \in M_h$ such that

$$\langle Tu_h, v_h - u_h\rangle + j(v_h) - j(u_h) \geq \langle A(u_h), v_h - u_h\rangle, \tag{4.1}$$

With these hypotheses and preliminaries established, we can now derive the following abstract error estimate.

THEOREM 4.1. *Let $u \in M$ and $u_h \in M_h$ be the solutions of (3.1) and (4.1) respectively. Let T be a strongly monotone and Lipschitz continuous operator. If A is a Lipschitz continuous antimonotone operator, and $Tu - A(u) \in U$, then there exists a constant $c > 0$ such that*

(i) For $M_h \not\subset M$

$$\|u-u_h\|_H \leq c\{\|u-v_h\|_H+\|v-u_h\|_H+\|v_h-v\|_H+(\|A(u)-Tu\|_U\|u-v_h\|_{U'}$$
$$+\|A(u_h) - T(u_h)\|_U\|u_h - v\|_{U'})^{\frac{1}{2}} + (\|v - u_h\|_H + \|v_h - u\|_H)^{\frac{1}{2}}\},$$

$$\text{for all } v \in M, v_h \in M_h \tag{4.2}$$

(ii) For $M_h \subset M$,

$$\|u - u_h\|_H \leq c\{\|u - v_h\|_H + (\|A(u) - Tu\|_U\|u - v_h\|_{U'})^{\frac{1}{2}} + (\|u - v_h\|_H)^{\frac{1}{2}}\},$$

$$\text{for all } v_h \in M_h \tag{4.3}$$

Proof. Since $u \in M$ and $u_h \in M_h$ are solutions of (3.1) and (4.1) respectively so by adding these inequalities, we have

$$\langle Tu, Tu\rangle + \langle Tu_h, Tu_h\rangle \leq \langle Tu, v\rangle + \langle Tu_h, v_h\rangle + \langle A(u), u - v\rangle$$
$$+ \langle A(u_h), u_h - v_h\rangle - j(u) - j(u_h) + j(v) + j(v_h).$$

Subtracting $\langle Tu, u_h\rangle + \langle Tu_h, u\rangle$. from both sides and rearranging terms, we get

(4.4)

$$\begin{aligned}\langle Tu - Tu_h, u - u_h\rangle \le \langle Tu - Tu_h, v - u_h\rangle + \langle Tu - Tu_h, u - v_h\rangle \\ + \langle A(u) - A(u_h), v_k - v\rangle + \langle A(u) - Tu, u - v_h\rangle \\ + \langle A(u_h) - Tu_h, u_h - v\rangle + j(v) - j(u_h) + j(v_h) = ju\end{aligned}$$

Since by assumption, $Tu - A(u) \in U$, T is strongly monotone Lipschitz continuous and A is Lipschitz continuous, so we obtain

$$\begin{aligned}\alpha\|u - u_h\|^2 \le \beta\|u - u_h\|(\|v - u_h\| + \|u - v_h\|_H) + \gamma\|u - v_h\|_U\|v_h - v\|_{U'} \\ + \|A(u) - Tu\|_U\|u - v_h\|_{U'} + \|A(u_h) - Tu_h\|_U\|u_h - v\|_{U'} \\ + \xi(\|v - u_h\|_H + \|v_h - u\|_H).\end{aligned}$$

Using the Young's inequality.

$$ab \le \in a^2 + \frac{1}{4\in}b^2,$$

for positive a, b, and $\in > 0$, we have the required result (4.5).

(ii) For $M_h \subset M$, we obtain (4.3) from (4.2), by taking $v = u_h$ and using the fact that $\|v_h - u_h\|_H \le v_h - u\|_H + \|u - u_h\|_H$.

5. Applications

The general class of contact problems considered here are characterized by the following system of equations and inequalities:

$$-\sigma_{ij}(u)_j = f_i(u), \quad \sigma_{ij}(U) = E_{ijkl}u_{k,l} \text{ in } \Omega;$$

$$u_i = 0 \text{ on } \Gamma_D; \quad \sigma_{ij}n_j = t_i \text{ on } \Gamma_F;$$

$$(5.1) \qquad \left.\begin{aligned} v_F|F_n| = g, \quad |\sigma_T| < g \Rightarrow u_T = 0, \\ |\sigma_T| = g \Rightarrow \Lambda \ge 0 \text{ such that } u_T = -\Lambda\sigma_T\end{aligned}\right\} \text{ on } \Gamma_C.$$

Using the techniques of Duvaut and Lions [6], Oden and Pires [1], one can derive the following variational principle characterizing problem (2.1).

Find a displacement field $u \in M$ such that

$$a(u, v - u) + j(v) - j(u) \geq \langle f'(u), v - u\rangle \quad \text{for all} \quad v \in M, \tag{5.2}$$

where $f'(u)$ is the Fréchet differential of the nonlinear differentiable functional

$$f(u) = \int_\Omega \int_0^\mu f(\eta) d\eta \, dx \quad \text{at} \quad u \in H.$$

Here the following notation and conventions are used:

$\Omega =$ the elastic body in a bounded open domain in $\mathbb{R}^N$ with Lipschitz boundary $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C$, where $\Gamma_D(\Gamma_F)$ are portions of Γ on which the displacements (tractions) are prescribed, and Γ_c is the (candidate) contact surface on which a body may contact the foundation upon application of the loads. Also, it is assumed throughout that $\bar{\Gamma}_C \cap \bar{\Gamma}_D = \phi, u = (u_1, u_2, \cdots, u_N)$ is the displacement vector, $u = u(x)$, where $x = (x_1, x_2, \cdots, x_N)$ is a point in Ω.

$$\langle Tu, v\rangle = \int_\Omega \sigma_{ij}(u) \in_{ij} (v) dx \tag{5.3}$$

=the virtual work produced by the stresses.

$\sigma_{ij}(u) = E_{ijkl} u_{k,l}$, corresponding to displacement u on the strains $\in_{ij}$ $(v) = 1/2(v_{i,j} + v_{j,i})$ due to the virtual displacement v. E_{ijkl} is Hooke's tensor of elastic constants satisfying the usual ellipticity and symmetry conditions.

$j(v) = \int_{\Gamma_C} g|g_T| ds$
= virtual work on contact surface due to frictional stresses.

$$f(v) = \int_\Omega \int_0^v f(\eta) d\eta \, dx + \int_{\Gamma_F} t_i v_i ds + \int_{\Gamma_C} F_n v_n ds. \tag{5.4}$$

In the expression for $g(\cdot)$, $f(\eta)$ is the body force per unit volume depending on the displacement field u, assumed to be given as a smooth function in $L_2(\Omega)$; t_i are components of surface traction, assumed to be given as functions in $L_\infty(\Gamma_F)$ and F_n is the prescribed normal contact pressure on Γ_C

$M =$ the constraint set corresponding to the unilateral condition $\Lambda(u) \cdot n \leq s = \{v \in H, \Lambda(v) \cdot n \leq s \text{ in } W\}$, where s is given in W, the space of normal traces of admissible displacements on Γ_C, see [1].

$\nu_F =$ the coefficient of friction ; $0 \leq \nu_F$.

S = the normalized initial gap between the body Ω and the foundation prior to the application of loads.

$u_n = u \cdot n$ = normal displacement of particles on the boundary Γ.

$\sigma_n(u)$ = normal contact pressure $= \sigma_{ij}(u)n_i n_j; n_i$ = components of unit outward norm to Γ.

$H = \{v \in (H'(\Omega))^{N_i}; \gamma(v) = 0$ a.e. on $\Gamma_D\}$, wherein γ is the trace operator mapping $H'(\Omega)$ onto $H^{1/2}(\Gamma)$, and v_T is the tangential component of v on Γ_C. Here and throughout, we use the notation and conventions commonly used in the study of partial differential equations and in the study of contact problems by variational methods, see [1].

REMARK 5.1. The variational inequality (5.2) characterizes the Signorini problem in elastostatics with friction forces. Inequality (5.2) is merely a statement of the principle of virtual work for an elastic body restrained by frictional forces. The strain energy of the body corresponding to an admissible displacement v is $1/2\ \langle Tv, v\rangle$ is the work produced by $\sigma_{ij}(u)$ through strains caused by the virtual displacemnent $v - u$. The form f depending upon the displacement u represents the work done by the external forces, and $j(\cdot)$ represents the work done by the frictional forces.

Problems of the type

$$-\sigma_{ij}(u)_j = f_i, \quad \sigma_{ij}(u) = E_{ijkl}u_{h,l} \text{ in } \Omega$$

$$u_i = 0 \text{ on } \Gamma_D; \quad \sigma_{ij}n_j = t_i \text{ on } \Gamma_F$$

$$(5.5) \qquad \left.\begin{array}{l} v_F|F_n| = g, \quad |\sigma_T| < g \Rightarrow u_T = 0, \\ |\sigma_T| = g \Rightarrow \lambda \geq 0 \text{ such that } u_T = -\lambda\sigma_T \end{array}\right\} \text{ on } \Gamma_C.$$

for which f_i is a function only of the space variables, have been studied by Duvaut and Lions [6]. In this case, Duvaut and Lions [6] have derived the following variational principle characterizing problem (5.4).

Find a displacement field $u \in M$ such that

$$(5.6) \qquad a(u, v - u) + j(v) - j(u) \geq \langle f, v - u\rangle \text{ for all } v \in M.$$

In order to derive the error estimate, we assume that the given function $f(u)$ is antimonotone, i.e.

$$(5.7) \qquad \int_\Omega (f(u) - f(v))(u - v)dx \leq 0$$

and

$$\|f(u)\|_{L_2(\Omega)} \leq r\{\|u\|_{H'}\}, \tag{5.8}$$

where $r = r(t)$ is a nondecreasing function for $t \in R$, $t > 0$. If the function $f(u)$ defined by (5.4) is Fréchet differentiable, then (see Noor and Whitemann [19]),

$$\langle f'(u), v\rangle = \int_\Omega f(u)v dx. \tag{5.9}$$

Using relation (5.7) - (5.9), one can easily show that $f'(u)$ is antimonote and

$$\|f'(u)\| \leq \|f(u)\|_{L_2(\Omega)} \leq r\{\|u\|_{H'}\}. \tag{5.10}$$

In order to apply the results of Sections 3 and 4, we must show that all the hypotheses of Theorem 3.1 and Theorem 4.1 are satisfied. Now, in view of the symmetry and ellipticity of E_{ijkl}, the operator T defined by (5.3) is strongly monotone and Lipschitz continuous. The functional $j(v) = \int_{\Gamma_C} g|v_T|ds$ is obviously nondifferentiable, convex, proper and lower semi-continuous. As the nonlinear function $f(u)$ defined by (5.4) is antimonotone and Lipschitz continuous by the assumptions, thus showing that all the hypotheses of Theorem 3.1 are satisfied. Hence it follows from Theorem 3.1 that there does exist a unique solution of (5.2).

FiniteElementApproximation

We now consider the finite element approximation of the variational inequality (3.1). Following standard finite element techniques, we construct a partition of Ω into a mesh of finite elements over which the displacements are approximated by piecewise polynomials. This defines a finite dimensional subspace S_h of H. By constructing a sequence of regular refinements of the mesh, we generate a family $\{S_h\}_{h>0}$, of subspaces of H. It is well known [1] that S_h exhibit the following asymptotic interpolation properties.

If $u \in (H^\Gamma(\Omega))^N, r \geq 0$, then there exists a constant $c > 0$ such that

$$\inf_{v_h \in S_h} \{\|u - v_h\|_0 + h\|u - v_h\|_1\} \leq Ch^s\|u\|_{r,\Omega}, s = \min(3, r-1), \tag{5.11}$$

Let Γ_C^h denote the boundary of the mesh that approximates Γ_C and $\sum_e$ denote the set of all nodal points e on Γ_C^h. We assume that $\sum_e = \Gamma_C^h \cap \Gamma_C$.

As an approximation of the constraint set M, we introduce

$$(5.12)\quad M_h = \{\phi_h \in C^0(\Gamma_V^h) : \\ \phi_h = \gamma(v)h)\cdot N, v_h \in S_h, \phi_h(e) \leq s_h(e), \text{for all } e \in \sum_e\}$$

where s_h is the $L_2(\Gamma_C)$-projection of e on the space W_h of normal traces of functions in S_h. Thus, in our discrete model of the friction problem, we impose the contact condition only at the nodal points on Γ_C^h. Clearly, in general, $M_h \not\subset M$.

We also need the following result, which can be easily proved by using the methods of Noor [20] and Janovsky and Whiteman [4].

LEMMA 5.1. *There exists a constant C_1 such that*

$$(5.13)\qquad \|f(u_h)\|_{L_2(\Omega)} \leq C_1, \qquad \text{for all } h > 0$$

For simplicity, we only consider the special case of Theorem 4.1 (ii), where $M_h \subset M$. Taking $U + U' = (L_2(\Omega))^2$ and using Lemma 5.1, we obtain

$$(5.14)\qquad \|u - u_h\|_1 = 0(h^{1/2}).$$

Note that in the absence of the friction term $b(u,v)$, it has been shown in [20], that the error estimate in the energy norm is of order h.

6. Concluding Remarks

In this study, we have only studied a class of nonlinear variational inequalities. The general theory discussed in this paper can be used to formulate variational principles for a wide range of free boundary problems. These include problems in elasticity, optimal control problems in the dynamics of distributed systems, interface problems, equilibrium problems in transportation and economics, and many others. It is true that each of these areas of applications requires special consideration of peculiarities of the physical problem at hand and the inequalities that models. In this paper, we have considered and studied a general auxiliary variational inequality problem. It is shown that the auxiliary principle can ve used to suggest an innovative and novel algorithm for computing the approximate solution of variational inequalities. The auxiliary problem proposed in this paper is quite general and flexible. By appropriate choice of the auxiliary problem, one may be able to select suitable method to solve the variational inequalities and related optimization problems. The auxiliary principle and suggested in this paper may be extended for the multivalued operators involving variational inequalities. We have also obtained the error estimates for the finite element approximations of nonlinear variational inequalities, which is of order $h^{1/2}$ in the energy norm. In this paper, we have merely described a class of variational inequalities. They are many fascinating and interesting applications of variational inequalities in many different branches of pure and applied sciences. We leave the exploration of these ideas to the interested reader. For related works, see [22-31] and the references therein.

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