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Preservation of some partial orderings of life distributions under length biased distributions

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ABSTRACT

For studies in reliability, biometry and survival analysis, the length biased distribution is frequently appropriate for certain natural sampling plans. So, we shall convey the preservation of some partial orderings under life length biasd distributions and closures of ILR and NBU classes under life length biasd distributions.

1. Introduction

For studies in reliability, biometry and survival analysis, the length biased distribution is frequently appropriate for certain natural sampling plans. The length biased distribution finds various applications in biomedical areas as family history and disease, early detection of disease survival and intermediate events and latency periods of AIDS due to blood transfusion, see Zelen and Feinleib[6].

Suppose that X and Y be nonnegative absolutely continuous random variables with probability density functions $f(x)$ and $g(x)$, respectively. Let F and G be the cumulative distributions of X and Y, $\overline{F}(x) = \int_x^\infty f(\mu)d\mu$ and $\overline{G}(x) = \int_x^\infty g(\mu)d\mu$ be the corresponding survival functions. Partial oderings,

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namely, likelihood ratio odering, weak likelihood ratio odering, failure rate ordering and stochastic odering between two random variables X and Y are known in the literature.

Gupta and Keating(1986) introduced relations for reliability measures under length biased sampling. Singh(1989) defined two new partial oderings and discussed relevances of these partial oderings for comparing life of a new unit with residual life of a used unit.

In this paper, we shall convey the preservation of some partial orderings under life length biasd distributions and closures of ILR and NBU classes under life length biasd distributions.

2. Preservation of some partial orderings of life distributions under length biased distributions.

For the sake of completeness, we briefly define some partial orderings which are known in the literature.

Definition 2.1 [4]. X is said to be larger than Y in likelihood ratio ordering, written as $X \geq^{LR} Y$, if $\frac{f(x)}{g(x)}$ is nondecreasing in $x \geq 0$.

Definition 2.2 [5]. X is said to be larger than Y in weak likelihood ratio ordering, written as $X \geq^{WLR} Y$, if $\frac{f(x)}{g(x)} \geq \frac{f(0)}{g(0)}$, where $\frac{f(0)}{g(0)}$ is assumed to belong to $(0, \infty)$.

Definition 2.3 [4]. X is said to be larger than Y in failure rate ordering, written as $X \geq^{FR} Y$, if $r_F(x) \geq r_G(x)$ for all $x \geq 0$, where $r_F(x)(= \frac{f(x)}{\overline{F}(x)})$ and $r_G(x)$ are failure rate functions of X and Y, respectively.

Definition 2.4 [4]. X is said to be larger than Y in stochastic ordering, written as $X \geq^{ST} Y$, if $\overline{F}(x) \geq \overline{G}(x)$ for all $x \geq 0$.

Let us now direct our attention at deriving expressions for the reliability measure, $\overline{F} \cdot (t)$, $r_F \cdot (t)$ and $e_F \cdot (t)$ for the corresponding survival function, failure rate function and the mean residual life fuction of length biaesd distribution version, where $e_F(x) = \frac{1}{\overline{F}(x)} \int_x^\infty f(\mu) d\mu$. And let $\mu_F = \int_0^\infty \overline{F}(\mu) d\mu$.

$$f \cdot (x) = \frac{x f(x)}{\mu_F}, x > 0,$$

$$\overline{F} \cdot (t) = -\int_x^\infty \frac{t \overline{F}'(t)}{\mu_F} dt = \frac{\overline{F}(x)\{x + e_F(x)\}}{\mu_F},$$

$$r_F \cdot (x) = \frac{f \cdot (x)}{\overline{F} \cdot (x)} = \frac{x r_F(x)}{\{x + e_F(x)\}},$$

and

$$\begin{aligned} e_F \cdot (x) &= \int_x^\infty \frac{\overline{F} \cdot (t)}{F \cdot (x)} dt \\ &= \int_x^\infty \frac{\overline{F}(t)\{t + e_F(t)\}}{\overline{F}(x)\{x + e_F(x)\}} dt \\ &= \frac{e_F(x)}{(x + e_F(x))} \int_x^\infty \frac{t + e_F(t)}{e_F(t)} exp\{-\int_x^t \frac{1}{e_F(\mu)} d\mu\} dt. \end{aligned}$$

Let $X\cdot$ and $Y\cdot$ be length biasd random variable corresponding of X and Y, respectively. First of all, we shall convey preservation of some partial orderings under length biasd distribution. Let $\frac{f \cdot (0)}{g \cdot (0)} = \lim_{x \to 0+} \frac{x f(x)/\mu_F}{x g(x)/\mu_G}$.

Theorem 2.1. $X \geq^{LR} Y$ if and only if $X\cdot \geq^{LR} Y\cdot$.

Proof.

$$\begin{aligned} X \geq^{LR} Y &\Longleftrightarrow \frac{f(x)}{g(x)} \quad \text{is nondecreasing in} \quad x \geq 0 \\ &\Longleftrightarrow \frac{f(x)/\mu_F}{g(x)/\mu_G} \quad \text{is nondecreasing in} \quad x \geq 0, \quad \text{for} \quad \frac{\mu_F}{\mu_G} > 0 \\ &\Longleftrightarrow \frac{f \cdot (x)}{g \cdot (x)} \text{ is nondecreasing in } x \geq 0 \\ &\Longleftrightarrow X\cdot \geq^{LR} Y\cdot \end{aligned}$$

Theorem 2.2. $X \geq^{WLR} Y$ if and only if $X\cdot \geq^{WLR} Y\cdot$.

Proof.

$$\begin{aligned} X \geq^{WLR} Y \iff & \frac{f(x)}{g(x)} \geq \frac{f(0)}{g(0)} \\ \iff & \frac{\mu_F}{\mu_G}\frac{f(x)}{g(x)} \geq \frac{\mu_F}{\mu_G}\frac{f(0)}{g(0)}, \quad \text{for} \quad \frac{\mu_F}{\mu_G} > 0 \\ \iff & \frac{f\cdot(x)}{g\cdot(x)} \geq \frac{f\cdot(0)}{g\cdot(0)} \\ \iff & X\cdot \geq^{WLR} Y\cdot. \end{aligned}$$

Theorem 2.3. If $X \geq^{FR} Y$, then $X\cdot \geq^{FR} Y\cdot$.

Proof. Suppose that $X \geq^{FR} Y$, then $r_F(x) \geq r_G(x)$. We using the fact that $r_F(x) \geq r_G(x)$ imply $e_F(x) \leq e_G(x)$.

$$r_F\cdot(x) = \frac{x r_F(x)}{x + e_F(x)} \leq \frac{x r_G(x)}{x + e_G(x)} = r_G\cdot(x).$$

Therefore $X\cdot \geq^{FR} Y\cdot$

Now, we consider next an example.

Example 1. We consider two survival functions $\overline{F}$ and $\overline{G}$. Suppose

$$\overline{F}(t) = \begin{cases} \exp(-t), & 0 \leq t < 1 \\ \exp(-\sqrt{t}), & t \geq 1. \end{cases}$$

and

$$\overline{G}(t) = exp(-\sqrt{t}), t \geq 0.$$

It is clear that $\overline{F}(t) \geq \overline{G}(t)$ with strictly inequality holds if and only if $0 < t < 1$. We observe that $\mu_F = \dfrac{e+3}{e}$ and $\mu_G = 2$. For $0 < x < 1$, we obtain the following.

$$\int_x^\infty \overline{F}(t)dt = \int_x^1 e^{-t}dt + \int_1^\infty e^{-\sqrt{t}}dt = e^{-x} + 3e^{-1}$$

$$\int_x^\infty \overline{G}(t)dt = 2(\sqrt{x}+1)e^{-\sqrt{x}}$$

and

$$\overline{F}\cdot(t)=\begin{cases}\frac{e}{e+3}\{(t+1)\exp(-t)+\frac{3}{e}\}, & 0\le t<1\\ \frac{e}{e+3}\exp(-\sqrt{t})\{t+2(\sqrt{t}+1)\}, & t\ge 1.\end{cases}$$

$$\overline{G}\cdot(t)=\frac{1}{2}\exp\left(-\sqrt{t}\{t+2(\sqrt{t}+1)\}\right),t\ge 0$$

Therefore, for $t\ge 1$, $\overline{F}\cdot(t)\le\overline{G}\cdot(t)$, i.e $X\cdot\le^{ST}Y\cdot$. It is counter example that even though X and Y are stochastic ordering, $X\cdot$ and $Y\cdot$ are not stochatic ordering.

Theorem 2.4. If $X\ge^{ST}Y$ and $EX=EY$, then $X\cdot\ge^{ST}Y\cdot$.

Proof. $\overline{F}\cdot(x)=-\int_x^\infty\frac{t\overline{F}'(t)}{\mu_F}dt=\frac{1}{\mu_F}[x\overline{F}(x)+\int_x^\infty\overline{F}(\mu)d\mu]$ and $\overline{G}\cdot(x)=\frac{1}{\mu_G}[x\overline{G}(x)+\int_x^\infty\overline{G}(\mu)d\mu]$. We using the fact that $\overline{F}(x)\ge\overline{G}(x)$ imply $\int_x^\infty\overline{F}(\mu)d\mu\ge\int_x^\infty\overline{G}(\mu)d\mu$. Hence $\overline{F}\cdot(x)\ge\overline{G}\cdot(x)$ that is $X\cdot\ge^{ST}Y\cdot$

3. Closures of classes of distributions under length biased distribution

We know that $r_F(x)\ge r_F\cdot(x)$, $\overline{F}\cdot(t)\ge\overline{F}(t)$ and $e_F\cdot(t)\ge e_F(t)$. And also know that if X is IFR(Increasing Failure Rate) then $X\cdot$ is also IFR. So, we consider closures of classes(ILR,NBU) of distributions under length biased distribution. First, we give two definitions which are known in the literature.

Definition 3.1 [4]. X is ILR (Increasing Likelihood Ratio) distribution if $X_s\ge^{LR}X_t$ for all $0<s\le t$.

Definition 3.2 [1]. X is New Better than Used (NBU) distribution if $\overline{F}(t+x)\le\overline{F}(x)\overline{F}(t)$ for $x\ge 0,t\ge 0$.

Theorem 3.1. X is ILR if and only if $X\cdot$ is ILR.

Proof. Using Theorem 2.1,

$$X \text{ is } ILR \Longleftrightarrow X \geq^{LR} X_t \quad \text{for all} \quad x \geq 0, \quad \text{where} \quad X_t = (X-t)|X>t$$

$$\Longleftrightarrow X\cdot \geq^{LR} X_t\cdot \quad \text{for all} \quad t \geq 0$$

$$\Longleftrightarrow X\cdot \quad \text{is} \quad ILR.$$

We consider a renewal process as generated by putting as item, with life distribution F, in use and the replacing it upon failure, the limiting distribution of residual life has distribution fuction $H_F(x) = \dfrac{1}{\mu_F}\displaystyle\int_0^x \overline{F}(u)du$, and density function $h_F(x) = \dfrac{\overline{F}(x)}{\mu_F}$. Let Z denote the random variable having density $h_F(x)$ and $Z_t = (Z-t)|Z>t$ for any $t \geq 0$. And let $Z\cdot$ denote the random variable having density $h_F\cdot(x)$ and $Z_t\cdot = (Z\cdot - t)|Z\cdot > t$ for any $t \geq 0$. Now, we consider closures of NBU classes under length biased distribution.

Theorem 3.2. X is NBU if and only if $X\cdot$ is NBU.

Proof. Using Theorem 4.1 of Singh[5] and theorem 2.2,

$$X \text{ is } NBU \Longleftrightarrow X \geq^{ST} X_t \quad \text{for all} \quad t \geq 0$$

$$\Longleftrightarrow Z \geq^{WLR} Z_t \quad \text{for all} \quad t \geq 0$$

$$\Longleftrightarrow Z\cdot \geq^{WLR} Z_t\cdot \quad \text{for all} \quad t \geq 0$$

$$\Longleftrightarrow X\cdot \geq^{ST} X_t\cdot \quad \text{for all} \quad t \geq 0$$

$$\Longleftrightarrow X\cdot \quad \text{is} \quad NBU.$$

4. Conclusion

The purpose of this paper is to consider preservation of some partial orderings of life distribution and closures of classes of life distributions under length biased distribution. We obtain the next results.

1) $X \geq^{LR} Y$ if and only if $X\cdot \geq^{LR} Y\cdot$.

2) $X \geq^{WLR} Y$ if and only if $X\cdot \geq^{WLR} Y\cdot$.

3) If $X \geq^{FR} Y$, then $X\cdot \geq^{FR} Y\cdot$.

4) If $X \geq^{ST} Y$ and $EX = EY$, then $X\cdot \geq^{ST} Y\cdot$.

5) X is ILR if and only if $X\cdot$ is ILR.

6) X is NBU if and only if $X\cdot$ is NBU.

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